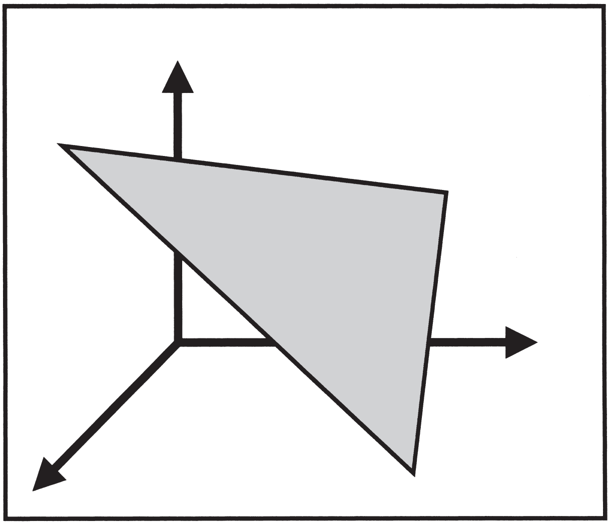
Erhard Rhyn

Trigonometry and Vector Geometry



Exercises **Answers**

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Translation to English: Michael Graf

Collection of exercises

Trigonometry and Vector Geometry

Answers

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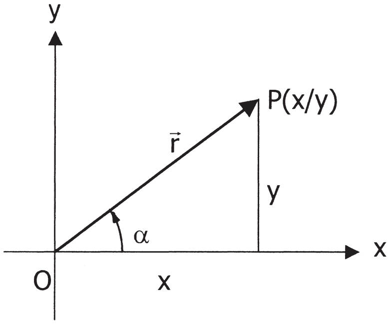
1. TRIGONOMETRY



$\sin \alpha = \frac{y}{r}$

$\cos \alpha = \frac{x}{r}$

$\tan \alpha = \frac{y}{x}$

$r = |\vec{r}| = \overline{OP}$

a) Introduction

1. Use a calculator to find
   a) the sine of 14°, 37.2°, 98°, 105.2°, 271.1°, 503°, -90°, -166.2°;
   b) the cosine of 3.2°, 81.4°, 178°, 293°, 720°, -100°, -28.6°;
   c) the tangent of 23°, 112.2°, 269.8°, 358°, 405°, -44.8°, -180°;
   d) the angle with the sine value of 0.8 , 0.31 , 0.99 , 0.001 , 0.43 ;
   e) the angle with the cosine value of 0.5 , 0.77 , 0.8 , -0.4 , -0.39 ;
   f) the angle with the tangent value of 0.2 , 2 , 22 , 222 , 1.04 .

2. The following angles are given in radians.
   Use a calculator to work out
   a) the sine of $\frac{\pi}{4}$, $\frac{2\pi}{3}$, $\frac{\pi}{12}$, 0.4 , 5.8 , -6.1 , -2.05 ,
   b) the cosine of $\frac{11\pi}{6}$, $\frac{\pi}{2}$, $\frac{3\pi}{5}$, 1.9 , 3.14 , -3 , -10 ,
   c) the tangent of $\frac{\pi}{6}$, $\frac{3\pi}{2}$, $\frac{5\pi}{9}$, 2.1 , 12.56 , -5.5 , -0.92.

3. The following angles are given in radians. Convert them to degrees.
   a) $\frac{\pi}{4}$ b) $\frac{5\pi}{6}$ c) 1 d) 0.54 e) 5.81 f) 20

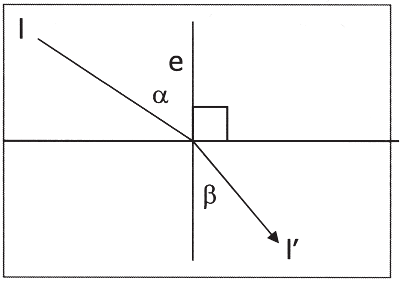
4. The following angles are given in degrees. Convert them to radians.
   a) 100° b) 444° c) 9.9° d) 707.03° e) 900°

5. Give the ranges of $\sin \alpha$, $\cos \alpha$, $\tan \alpha$.

6. Find the exact value of $\sin \alpha$, $\cos \alpha$, $\tan \alpha$ without using a calculator for
   $\alpha =$ a) 0°, b) 30°, c) 45°, d) 60°, e) 90°, f) 180°.

A ray of light hitting the boundary-plane of two optical media changes its direction.
According to Snells law (Snellius, 1591-1626), the following applies:

1. I, I' and e lie in one plane,
2. $\frac{\sin\alpha}{\sin\beta} = n = \text{constant}.$



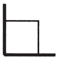
I ray of light
I' bent ray of light
e axis of incidence
α angle of incidence
β angle of refraction

At the transition of light from air into

- water, $n = {}^4/_3$;
- ordinary glass, $n = 1.54$.

7. Find the angle of refraction at the transition air / water, which corresponds to the following angle of incidence:
   a) 8°, b) 27°, c) 54.8°, d) 78.7°, e) 86°.

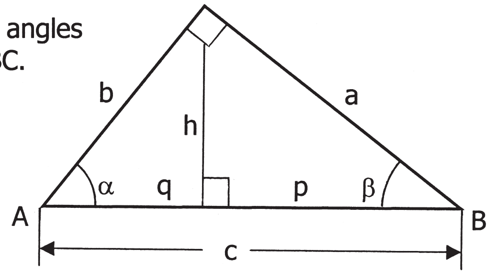
8. Find the angle of incidence at the transition air /glass, when the angle of refraction is
   a) 2°, b) 17.8°, c) 39.39°, d) 40.05°, e) 40.49°.

☞ In this book we use the following symbol to denote right angles: 

b) Right - angled triangles

9. Evaluate the missing sides and angles in the right - angled triangle ABC.

   a) a = 20, b = 21
   b) a = 88, c = 137
   c) a = 12, α = 40°
   d) b = 5.8, β = 78.2°
   e) c = 32.7, β = 47.3°
   f) c = 7.68, α = 3 β



10. Evaluate the missing sides and angles in the right - angled triangle ABC.

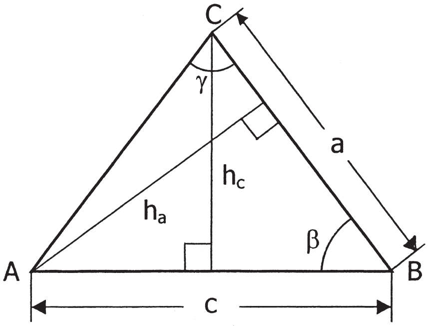
a) h = 12.3, b = 18.5	b) h = 6.08, α = 23.7°
c) p = 28, q = 63	d) a = 12.5, p = 4.4
e) p = 4.93, β = 70.3°	f) h = 9.1, q = 6.0

11. Similarly:

a) a = 27.8,	area	A = 373
b) a : b = 3 : 2,		A = 15
c) b : c = 5 : 13,	perimeter	p = 75
d) a : b = 3 : 4,	perimeter	p = 60

12. In the right-angled triangle ABC (hypotenuse $c = \overline{AB}$) are given:
   - a) a = 24, c = 74; ➪ find angle bisectors w_α and w_β;
   - b) h = 25, angle bisector $w_\gamma = 32$; ➪ find a and b;
   - c) a = 15, angle bisector $w_\beta = 17$; ➪ find median s_a;
   - d) b = 83, angle bisector $w_\alpha = 100$; ➪ find c;
   - e) $\alpha = 36°$, angle bisector $w_\alpha = 20$; ➪ find angle bisector w_β;
   - f) c = 39, $\beta = 52°$; ➪ find h_c, the altitude from C.

13. In the isosceles triangle ABC, find the missing sides and angles:
   - a) a = 25, c = 14;
   - b) c = 15, $\beta = 63°$;
   - c) a = 40.3, $h_a = 11.5$;
   - d) c = 7.38, $h_c = 8.76$;
   - e) $h_a = 34.2$, $\gamma = 51°$;
   - f) $h_c = 57.1$, $\gamma = 57.2°$.



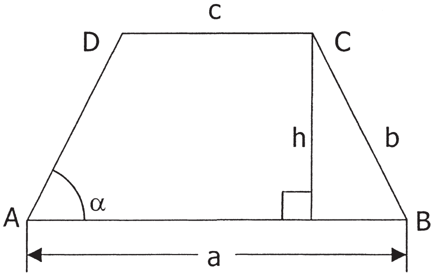
14. In a right-angled triangle ABC with hypotenuse c = 97 and other side b = 65, find the length of the angle bisector of the smallest interior angle.

15. In a rectangle with sides a = 2.8 and b = 4.5, find the acute angle at the intersection of the diagonals.

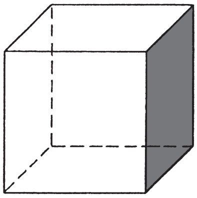
16. Find the size of the interior angles of a rhombus with diagonals e = 57.2 and f = 81.7.

17. A rhombus, one of whose interior angles is 61°, has an area A = 28. Find the lengths of the diagonals.

18. In the isosceles trapezoid illustrated find the missing quantities, given the following:
   - a) a = 24, b = 9, $\alpha = 64.8°$;
   - b) b = 61, c = 37, h = 17;
   - c) a = 40, b = 27, c = 29;
   - d) c = 29, h = 14, $\alpha = 71.5°$.

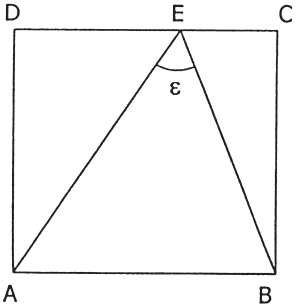


19. Given an isosceles trapezoid ABCD with parallel sides a = 45, c = 33 and diagonal e = 89, find the size of the base angles.

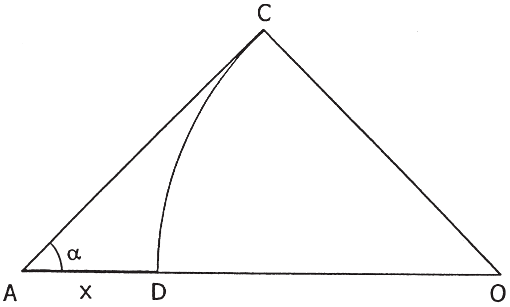


20. In a cube, find the angle between
    a) body diagonal and edge,
    b) face diagonal and edge,
    c) body diagonal and face diagonal.

21. Given $\overline{AB} = 6$, $\overline{DE} = 4$ of the square ABCD, find the size of angle ε.



22. Given $\alpha = 50°$, $\overline{AC} = \overline{CO} = \overline{DO} = 7$, find $x = \overline{AD}$.

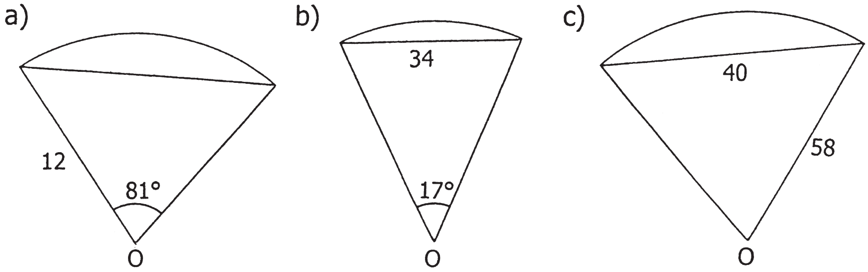


23. Let r be the radius of a circle. Find the central angle which corresponds to the chord s.
    a) r = 14, s = 21 b) r = 52.45, s = 103.66

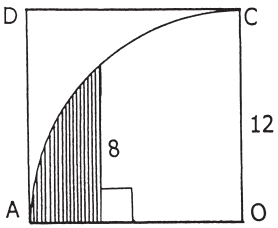
24. Consider a regular pentagon with sides of unit length. Find the area of the pentagon as well as the length of its diagonals.

25. A circle of radius r = 20 has a regular 15 - gon
    a) inscribed, b) circumscribed.
    Find the area of the 15 - gon in each case.

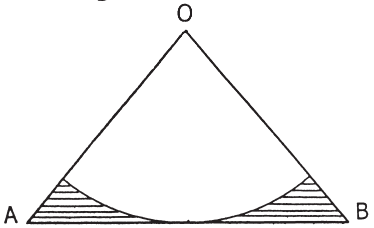
26. Find the area of the circular segment.



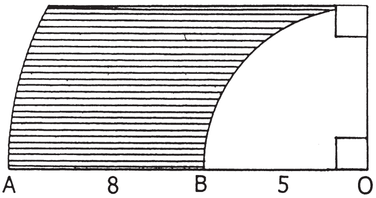
27. The quadrilateral AOCD is a square. What is the area of the shaded region ?



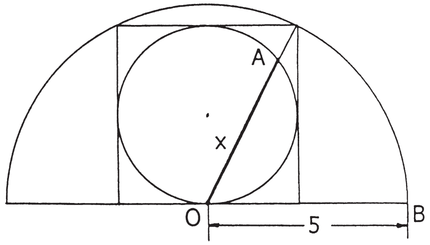
28. $\overline{AB} = 110$, $\overline{AO} = \overline{BO} = 73$. What is the area of the shaded region ?



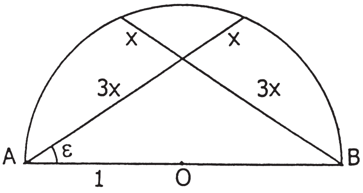
29. Find the area of the shaded region. O denotes the centre of both arcs.



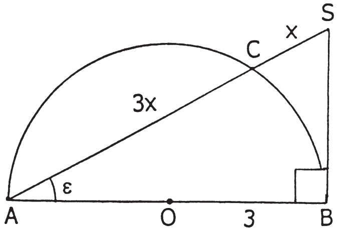
30. Semi-circle ⇨ square ⇨ circle; find $x = \overline{AO}$.



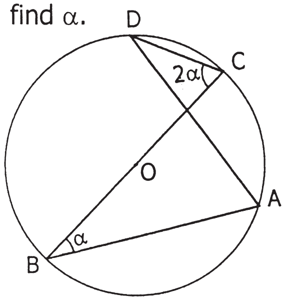
31. Find ε and x.



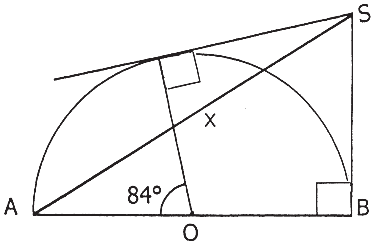
32. Find ε and x.



33. Radius $r = \overline{BO} = 5$, $\overline{AD} = 8$; find α.



34. Radius $r = \overline{AO} = 5$; find $x = \overline{AS}$.



35. A point is given by its rectangular coordinates x and y.
Calculate its polar coordinates r and φ.

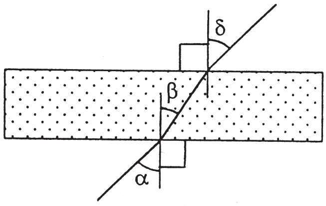
a) $x = 0, y = 3$ b) $x = 7, y = 0$ c) $x = -4, y = 0$
d) $x = 5, y = 5$ e) $x = -4, y = 4\sqrt{3}$ f) $x = 5, y = -12$
g) $x = -3, y = 7$ h) $x = -6, y = -8$ i) $x = -7, y = 24$

36. A point is given by its polar coordinates r and φ.
Find its rectangular coordinates x and y.

a) $r = 5, \varphi = 0°$ b) $r = 3, \varphi = 270°$ c) $r = 4\sqrt{2}, \varphi = 225°$
d) $r = 4, \varphi = 120°$ e) $r = 3, \varphi = 240°$ f) $r = 5, \varphi = 103°$
g) $r = 2, \varphi = 276°$ h) $r = 7, \varphi = 258°$ i) $r = 15, \varphi = 345°$

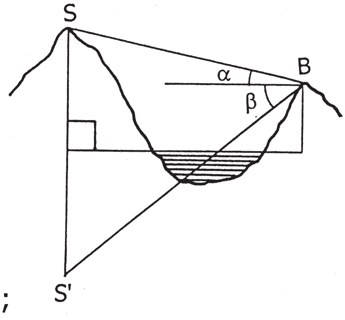
37a) The angle of elevation of the sun is 35°. A lamppost casts a shadow of 50 m onto a soccer field. Find the height of the post.
b) A lamppost 15 m high casts a shadow of 37 m. Find the angle of elevation of the sun.

38. A ray of light penetrates a glass top which is 35 mm thick. The angle of incidence α is 53.8° (⇨ see page 2)



a) What is the size of β and δ ?
b) What is the length of the ray's path within the glass top ?

39. An observer B is situated 70 m above sea level. The angle of elevation to the top of a mountain is $\alpha = 28°$, its mirror image S' is seen under the angle of depression of $\beta = 35°$. Find the height of S above sea level.



40. Find the height of a tower:
Horizontal distance of observer – tower 25 m;
Angle of elevation observer - spire 53°;
Angle of depression observer – base of tower 17°.

41. A satellite is stationed 100 km above the atlantic.
a) Find the angle of depression to the edge of the visible earth.
b) Calculate the distance to the edge of the visible earth.
(Assume the radius of the Earth is 6370 km; ignore the refraction of light in the atmosphere.)

c) Sine and cosine rules

42. For each of the quadrants show that the following are true:

2. quadrant	3. quadrant	4. quadrant
$\sin(180°-\alpha) = \sin\alpha$	$\sin(180°+\alpha) = -\sin\alpha$	$\sin(360°-\alpha) = -\sin\alpha$
$\cos(180°-\alpha) = -\cos\alpha$	$\cos(180°+\alpha) = -\cos\alpha$	$\cos(360°-\alpha) = \cos\alpha$
$\tan(180°-\alpha) = -\tan\alpha$	$\tan(180°+\alpha) = \tan\alpha$	$\tan(360°-\alpha) = -\tan\alpha$

43. Find all the solutions to the following equations where $0° \le \alpha \le 360°$:

a) $\sin\alpha = 0.2$ b) $\sin\alpha = -0.74$ c) $\cos\alpha = 0.84$
d) $\cos\alpha = -0.05$ e) $\tan\alpha = 21$ f) $\tan\alpha = -0.51$
g) $\cos\alpha = -0.9$ h) $\tan\alpha = -2$ i) $\sin\alpha = 0.27$
k) $\tan\alpha = 1.2$ l) $\cos\alpha = 0.003$ m) $\sin\alpha = -0.07$

44. In the triangle ABC, find the missing sides and angles:
a) $a = 15$, $\alpha = 35°$, $\beta = 50°$ b) $b = 8.2$, $\alpha = 54°$, $\gamma = 39°$
c) $c = 12$, $\beta = 9.7°$, $\gamma = 93.8°$ d) $a = 53$, $\alpha = 8°$, $\gamma = 126°$

45. Similarly:
a) $a = 20$, $b = 34$, $\beta = 72°$ b) $b = 9.3$, $c = 7.8$, $\beta = 51.3°$
c) $a = 42.8$, $c = 38.1$, $\gamma = 17.9°$ d) $a = 49$, $b = 57$, $\alpha = 84°$

46. Similarly:
a) $a = 8$, $b = 5$, $\gamma = 75°$ b) $a = 9.81$, $c = 7.25$, $\beta = 51.3°$
c) $b = 63.2$, $c = 31.1$, $\alpha = 109.3°$ d) $a = 67.3$, $c = 3.7$, $\beta = 176.3°$

47. Similarly:
a) $a = 5$, $b = 6$, $c = 7$ b) $a = 23$, $b = 38$, $c = 29$
c) $a = 25.7$, $b = 46.1$, $c = 36.8$ d) $a = 8.12$, $b = 15.66$, $c = 5.89$

48. Find the circumradius and area of the triangle ABC.
a) $b = 12$, $c = 17$, $\gamma = 70°$ b) $a = 35$, $b = 41$, $c = 22$

49. Find the angles of the acute triangle ABC, if $b = 2.78$, $c = 9.31$ and its area $A = 12.8$.

50. A triangle ABC with $a = 23$ and $\gamma = 59°$ has a circumradius $r = 21$. Find the missing values for α, β, b and c.

51. In the triangle ABC, the following are given:

a) $a = 5$, $b = 9$, $\gamma = 84°$; ⇨ find: angle bisector w_β;

b) $b = 57$, $c = 44$, $\beta = 61°$; ⇨ find: altitude h_c;

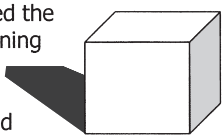
c) $a = 8.52$, median $s_a = 12.45$, $\gamma = 98.3°$; ⇨ find: c;

d) $a = 108$, angle bisector $w_\beta = 59$, $\beta = 61°$; ⇨ find: b.

52. Find the length of the angle bisector w_α in the isosceles triangle ABC, if the base $c = 92$ and $\alpha = \beta = 48°$.

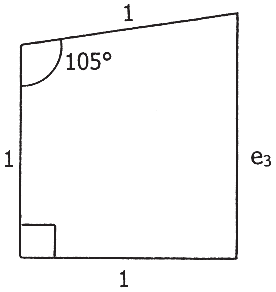
53. An isosceles triangle is given by $a = b = 9$ and $\alpha = \beta = 37°$. Find altitude h_a, median s_a and angle bisector w_α.

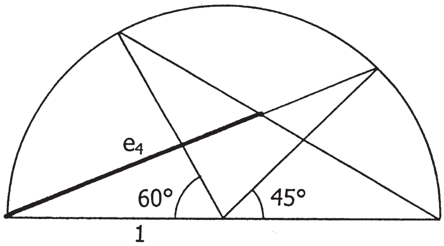
In the temple of Apollo on the island Delos, there used to be a cubic-shaped altar. When the inhabitants of Delos consulted the oracle in order to find out what could be done to avert a threatening plague epidemic, Apollo answered, "Build a cubic-shaped altar, twice the volume of the present altar." An exact construction of the new length of the edges (with just compass and ruler) turned out to be impossible.



54a) What is the length of the edge e_2 of the new cube, if the edge e_1 of the original cube has length 1 ?

b) For the construction of e_2, several approximations exist; two of them are illustrated below. Find e_3 and e_4.





55. In the triangle ABC, find the length of the three altitudes.

a) $a = 12$, $b = 17$, $c = 19$ b) $b = 45.8$, $\alpha = 57°$, $\gamma = 102°$

56. Consider the triangle ABC with $a = 8$, $b = 7$ and $\alpha = 75°$. The perpendicular bisector m_a intersects the altitude h_c at the point S.

a) Find the acute angle of intersection of m_a and h_c.

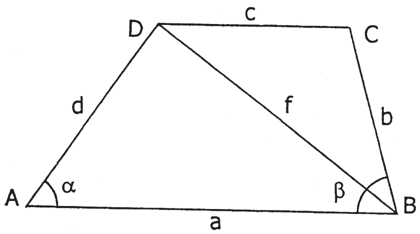
b) What is the length of $\overline{CS}$?

57. Find the inradius r of the triangle ABC, if b = 46.5, c = 21.2 and $\beta = 87.3°$.

58. Find the interior angles of the parallelogram ABCD where sides a = 83 and b = 109 and diagonal f = $\overline{BD}$ = 42.

59. Consider the parallelogram ABCD with a = 46, b = 18 and $\alpha = 36°$. Find the diagonals e and f.

60. Work out the missing quantities in the trapezoid ABCD.



a) a = 50, c = 20, d = 27, $\alpha = 71°$

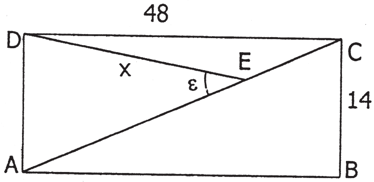
b) b = 19, c = 33, $\alpha = 47°$, $\beta = 59°$

c) b = c = d = 7.8, $\beta = 37°$

d) b = d = 12, f = 27, $\alpha = 70°$

61. What is the area of the isosceles trapezoid ABCD with a = 9.2 , b = d = 5.7 and $\alpha = 78°$?

62.

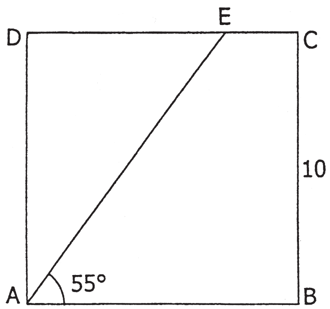


In the rectangle ABCD: $\overline{CE}$ = 10.

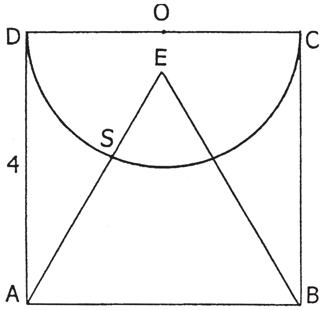
a) Find x = $\overline{DE}$.

b) Find the size of ε.

63. Consider the square ABCD. Point F lies on the line segment $\overline{AE}$ such that the triangles ABF and BCF have the same area. What is the distance between A and F ?

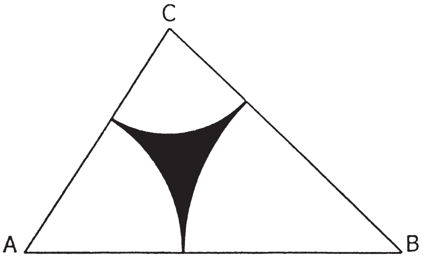


64. ABCD is a square and ABE an equilateral triangle. What is the distance between S a) and E, b) and D ?

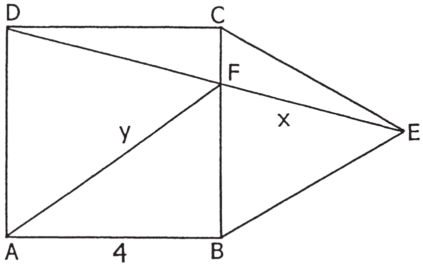


65. Given that: $\overline{AB} = 10$,
$\overline{BC} = 13$,
$\overline{AC} = 9$

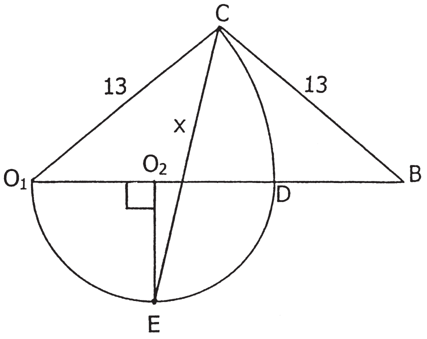
and that the arcs around A, B and C touch, find the area of the shaded region.



66. ABCD is a square, BEC an equilateral triangle
Find $x = \overline{EF}$ and $y = \overline{AF}$.

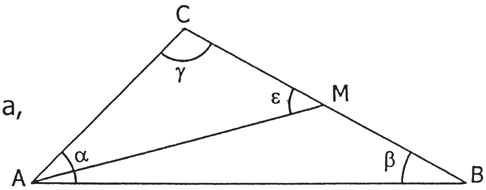


67. Consider the isosceles triangle O_1BC.
Its base is $\overline{O_1B} = 24$.
Find $x = \overline{CE}$.



68. In the triangle ABC, the medians $s_a = 12$ and $s_b = 9$ form a right angle. Find the angles of the triangle ABC.

69. In the triangle ABC:
M is midpoint of the side a,
$\alpha = \varepsilon = 45°$.
Find β and γ.

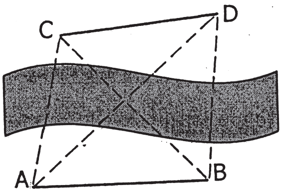


Finally, some concrete applications.

70. To measure the width of a river, a line segment $\overline{AB} = 30$ m is marked out along one side. On the opposite bank there is a tree C. From A and B, the angles $\measuredangle$ BAC = 68.3° and $\measuredangle$ CBA = 82.1° are measured.
What is the width of the river at C ?

71. The distance between A and an inaccessible point C is to be determined. To this end, a base $\overline{AB} = 127$ m is marked out and the angles $\measuredangle$ BAC = 58.3° and $\measuredangle$ CBA = 75.1° are measured. Find $\overline{AC}$.

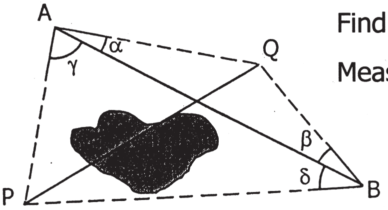
72.



The distance $\overline{CD}$ is to be determined from the other side of the river.

The following measurements are taken:
$\overline{AB} = 208$ m, $\measuredangle BAC = 71.4°$, $\measuredangle BAD = 55.6°$, $\measuredangle DBA = 93.0°$, $\measuredangle CBA = 49.3°$. Find $\overline{CD}$.

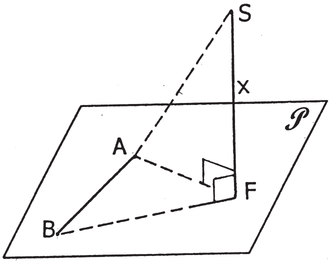
73.



Find the distance $\overline{PQ}$.

Measurements: $\overline{AB} = 380$ m, $\alpha = 41°$, $\beta = 77°$, $\gamma = 82°$, $\delta = 34°$.

74.



The mountain peak S is x meter above the horizontal plane $\mathcal{P}$.

Measurements: $\overline{AB} = 120$ m, $\measuredangle BAF = 48°$, $\measuredangle FBA = 76°$, $\measuredangle FAS = 71°$.

Find $x = \overline{FS}$.

75. A satellite is currently at an altitude of 100 km. An observer finds that the zenith angle to this satellite is 50°. Find the distance between the observer and the satellite at that given moment.

76. A ship passes point A. 20 minutes later, another ship passes the same point. The first ship sails with a course N35E (north 35° east) and a velocity of 24 knots, the second with N54W at 17 knots.
Supposing that the ships maintain their course and velocity, how far apart are they one hour after the second ship passes point A ?

☞ *1 knot: 1 nautical mile per hour; 1 nautical mile: 1.852 km*

77. A ship is seen by the coastguard in the direction N17E at a distance of 18 km and 15 minutes later at a distance of 23 km in the direction of N14W.
 a) What distance has the ship covered ?
 c) What is the course of the ship ?

78. In order to determine its position, a ship takes a bearing on a lightship in the direction of N17W and a lighthouse in the direction of N36E. According to the nautical chart, the lightship and the lighthouse are 16.4 nautical miles apart. The lighthouse, sighted from the lightship, is in direction N96E. Find the distance between the ship and the lightship.

79. An aeroplane follows the course N32E with an airspeed of 400 km/h. A southwesterly wind blows at a velocity of 80 km/h. Find the flying speed and the actual direction of flight.

d) Trigonometric curves

Sketch the graphs of the following functions for $0 \le x \le 2\pi$.
Find both the maximum and minimum value of the curve and the points of intersection with the coordinate axes.

80a) $y = \sin x$ b) $y = \cos x$ c) $y = \tan x$

81a) $y = \sin 2x$ b) $y = 4 \sin x$ c) $y = 3 \cos 2x$ d) $y = 1+\sin \frac{x}{2}$

82a) $y = \sin (x + \frac{\pi}{4})$ b) $y = 3 \cos (2x - \frac{\pi}{2})$

83. Sketch the graphs of the functions by superposition
a) $y = \sin x + \cos x$, b) $y = \sin x + \sin 2x$.

84. Find the values of x $(0 \le x \le 2\pi)$ for which the following functions are not defined.
a) $y = \frac{1}{\cos x}$ b) $y = \frac{1}{\sin 3x}$ c) $y = \frac{1}{1 + \sin x}$
d) $y = \frac{1}{\tan x}$ e) $y = \frac{1}{\cos^2 x - 1}$ f) $y = \frac{1}{1 + \tan x}$

85. Find the period of the functions (in radian).
a) $y = \sin 5x$ b) $y = 2 \cos 9x$ c) $y = \tan 8x$
d) $y = \cos \frac{3x}{4}$ e) $y = \sin \frac{2x}{3}$ f) $y = \tan \frac{x}{4}$

86. Find the range of the functions of exercise 85.

87. Find the values of x $(0 \le x \le 2\pi)$, which satisfy the following inequalities.

a) $\sin x < 0$ b) $\cos x > \frac{1}{2}$ c) $\tan x < 1$

d) $\cos x < \frac{\sqrt{2}}{2}$ e) $\tan x > \sqrt{3}$ f) $\sin x > \frac{\sqrt{2}}{2}$

e) Trigonometric identities

88. Prove the following identities.

a) $\sin^2\alpha + \cos^2\alpha = 1$ b) $\tan\alpha = \frac{\sin\alpha}{\cos\alpha}$ c) $1 + \tan^2\alpha = \frac{1}{\cos^2\alpha}$

89. Simplify the following terms.

a) $\frac{\sin\alpha}{\tan\alpha}$ b) $\frac{\cos\alpha}{\sin\alpha}$ c) $\tan\alpha \cdot \cos\alpha$ d) $\sin^4\alpha - \cos^4\alpha$

90. In each of the following you are given one of $\sin\alpha$, $\cos\alpha$ or $\tan\alpha$. Find the other two with the help of the identities in exercise 88 *without* determining the acute angle α.

a) $\sin\alpha = 0.28$ b) $\cos\alpha = \frac{12}{13}$ c) $\tan\alpha = \frac{3}{4}$

d) $\sin\alpha = \frac{15}{17}$ e) $\cos\alpha = \frac{99}{101}$ f) $\tan\alpha = \frac{35}{12}$

91. Determine the three trigonometric functions of $\alpha+\beta$ and $\alpha - \beta$, *without* finding α and β (assume α and β are acute angles).

a) $\sin\alpha = \frac{3}{5}$, $\sin\beta = \frac{9}{41}$ b) $\cos\alpha = \frac{8}{17}$, $\cos\beta = \frac{5}{13}$

92. Show that

a) $\cos(\alpha+\beta) + \cos(\alpha - \beta) = 2\cos\alpha\cdot\cos\beta$,
b) $\cos(\alpha - \beta) - \cos(\alpha+\beta) = 2\sin\alpha\cdot\sin\beta$,
c) $\sin(45° + \alpha) = \cos(45° - \alpha)$.

93. Simplify the following terms.

a) $\sin(\alpha+\beta) + \sin(\alpha - \beta)$ b) $\sin(\alpha+\beta) - \sin(\alpha - \beta)$ c) $\frac{\cos(\alpha-\beta)}{\cos\alpha\cdot\cos\beta}$

d) $\frac{\cos 2\alpha}{\cos^2\alpha}$ e) $\sin 2\alpha \cdot \tan\alpha$ f) $2\sin^4\alpha + \sin^2 2\alpha + 2\cos^4\alpha$

g) $\sin(60° + \alpha) - \sin(60° - \alpha)$ h) $\cos(60° + \alpha) + \cos(60° - \alpha)$

94. Evaluate, *without* determining α,

a) $\sin 2\alpha$ from $\sin \alpha = 0.6$;
b) $\cos 2\alpha$ from $\cos \alpha = 0.7$;
c) $\tan 2\alpha$ from $\tan \alpha = 3$;
d) $\sin 3a$ from $\sin \alpha = 0.1$;
e) $\cos 2\alpha$ from $\sin \alpha = 0.28$;
f) $\sin 2\alpha$ from $\tan \alpha = \sqrt{3}$.

95. Find the range of values of a for which the following equations are defined.

a) $\sin x = \frac{a}{3}$
b) $\sin x = \frac{3-a}{4}$
c) $\cos x = \frac{5}{a}$
d) $\cos x = \frac{a-2}{a+3}$
e) $\sin x = \frac{4a}{5}$
f) $\cos x = \frac{2a-1}{3a}$

☞ *For the following equations, give all solutions in the interval $0° \leq x < 360°$*

96a) $\sin (x + 10°) = \tan 40°$
b) $\cos (x - 9°) = -\sin 81°$
c) $\tan (315° - x) = 2 \cos 7°$
d) $3 \sin (x + 18°) = 4 \cos 56°$
e) $\tan (2x - 15°) = 7 \sin 265°$
f) $5 \cos (213° - x) = 2 \tan 118°$

97a) $\cos 2x = 0$
b) $\sin 3x = 0$
c) $\tan x = 1$
d) $\sin (x + 27°) = 0$
e) $\cos (2x - 15°) = 1$
f) $\tan (x + 58°) = 0$

98a) $\sin x - \cos x = 0$
b) $\sin x \cdot \cos x = 0$
c) $\frac{\sin x}{\cos x} = 0$

99a) $\sin x = 0.7 \cos x$
b) $\sin x = 3 \cos x$
c) $5 \sin x = 7 \cos x$

100a) $2 \sin x + \sin 2x = 0$
b) $\tan x + \sin x = 0$
c) $\tan^2 x - \tan x = 0$
d) $2 \sin^2 x - \sin x = 0$

101a) $\cos^2 x - \cos x = 0.75$
b) $\sin^2 x + \sin x = 0.75$

102a) $2 \cos x + \cos 2x = 0$
b) $\cos 2x + \sin^2 x = 0$
c) $\sin 3x + \sin x = 0$
d) $\cos 3x + \cos x = 0$

103a) $\sin x \cdot \cos x \cdot \tan x = 0.25$
b) $2 \tan x + \tan 2x = 0$

104a) $2 \cos^2 x + \cos x - 1 = 0$
b) $2 \tan^2 x + \tan x - 3 = 0$
c) $2 \sin^2 x + \sin x - 6 = 0$
d) $3 \cos^2 x + 5 \cos x - 2 = 0$

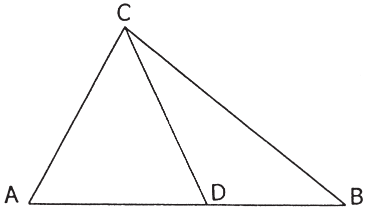
105a) $\cos^2 x = 1.5 \sin^2 x$
b) $3 \sin^2 x - \cos^2 x = 0$

106a) $\cos^2 x - 2 \sin x + 2 = 0$
b) $5 \cos^2 x + 7 \sin x + 7 = 0$

f) Trigonometric exercises for the Matura-exam

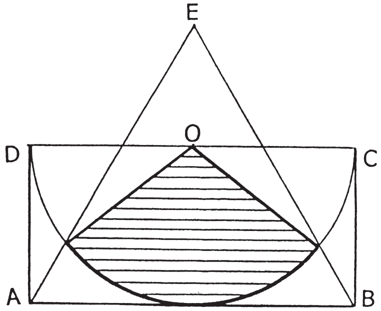
107. In the triangle ABC $a = 15$ cm, $b = 10$ cm and $c = 16$ cm. The triangles ADC and DBC have the same perimeter.

Find both the perimeter and the area of the triangle ADC.



108. In the rectangle ABCD, $\overline{AD} = 10$. The semicircle with diameter $\overline{CD}$ touches the line segment $\overline{AB}$. The triangle ABE is equilateral.

Find the area of the shaded sector.



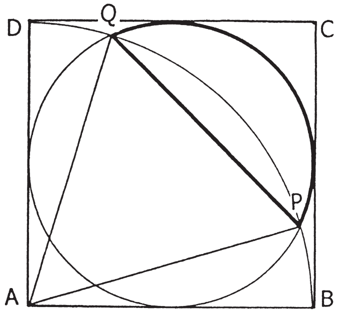
109.

The circumradius of this regular star is 10 cm. Its area is half that of the circum-circle.

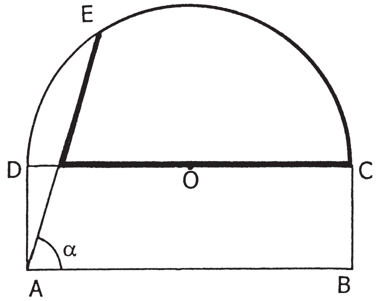
a) Find the perimeter of the star.
b) Find the size of the interior angles of the star.

110. The sides of a square ABCD have length $s = 6$ cm. The incircle of the square intersects a quadrant, with centre A and radius $\overline{AD}$, at the points P and Q.

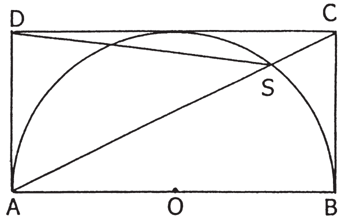
a) What is the size of the angle $\measuredangle$ PAQ ?
b) Find the area of the marked segment.



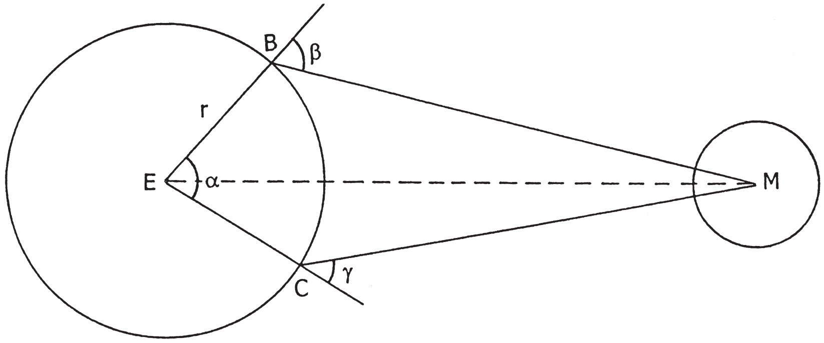
111. Consider the rectangle ABCD where $\overline{AB} = 30$, $\overline{BC} = 8$ and $\alpha = 75°$.
Above the line segment $\overline{CD}$, a semicircle is drawn.
a) Find the length of $\overline{AE}$.
b) Find the area of the marked region.



112. A semicircle is inscribed within the rectangle ABCD which has $\overline{AB} = 8$ and $\overline{BC} = 4$. (centre O, radius $\overline{OA}$). The diagonal $\overline{AC}$ intersects the semicircle at S.
a) Find the sides and angles of the triangle ASD.
b) What percentage of the total area of the rectangle is the area of the triangle?

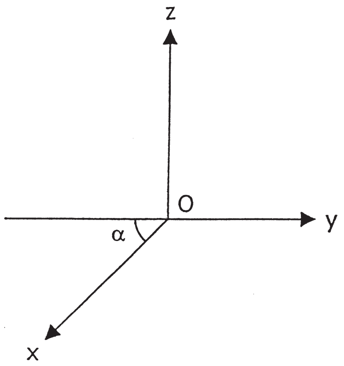


113. In 1771, French astronomers determined the distance between the moon and the earth using trigonometry. In order to do this, angle measurements were simultaneously taken in Berlin and Cape town (both towns lie almost on the same meridian):
$\alpha \approx 86.4411°$ (α is the sum of the latitudes of Berlin and Cape town),
$\beta \approx 41.2622°$, $\gamma \approx 46.5603°$.
Radius of the earth: $r \approx 6370$ km.
Determine the distance $\overline{EM}$ between the centres of the earth and the moon by means of these measurements.



2. OBLIQUE PARALLEL PROJECTION
- a graphic introduction to the geometry of space

The following indications are valid for **all** exercises in this chapter.



a) The constructions are to be drawn on A4 - paper.

b) The angle of contraction α is 45°, the scale of contraction is ½.

c) The detail (↓p, →q, ↕r) indicates the space required for this exercise. p, q, r: The origin of the coordinate system should be p cm away from the upper edge of the sheet and q cm from the edge to the left. r indicates the space required (vertically).

1. The points A and B are given.
   Sketch the segment $\overline{AB}$ in oblique parallel projection, in horizontal, vertical and side projection. Find the trace points of the line AB.

a) A(8/1/3),	B(4/3/1)	(↓4, →5, ↕10)
b) A(4/3/2),	B(2/6/3)	(↓5, →7, ↕11)
c) A/8/-2/5),	B(2/1/2)	(↓6, →6, ↕10)

☞ *trace points: points of intersection of the line with the coordinate planes.*

2. With the help of trace points, determine the visible part of the line AB.

a) A(10/-2/6),	B(2/6/2)	(↓4, →7, ↕11)
b) A(-3/6/3),	B(3/4/1)	(↓5, →7, ↕15)

3. Draw the straight line that is parallel to AB and that goes trough point C. Determine the visible part of both AB and the line through C.

a) A(6/4/0),	B(0/6/6),	C(6/1/-3)	(↓7, →3, ↕13)
b) A(3/3/2),	B(5/9/-2),	C(5/-3/4)	(↓4, →6, ↕9)

4. Shade the visible part of the triangle ABC.

a) A(7/5/-3),	B(4/8/6),	C(1/3/3)	(↓6, →4, ↕13)
b) A(6/-2/4),	B(10/6/-4),	C(2/6/4)	(↓5, →5, ↕14)
c) A(2/-3/4),	B(5/6/-2),	C(-2/9/-4)	(↓5, →5, ↕10)

5. Three edges of a cube with side length 6 lie on the positive coordinate axes. The line AB intersects it. Shade the visible part of the cube and the line AB.

   a) A(12/-3/-2), B(0/9/6) (↓7, →8, ↕14)
   b) A(2/4/7), B(12/-1/2) (↓9, →6, ↕14)
   c) A(5/12/-3), B(1/0/9) (↓10, →4, ↕16)

6. In a parallelogram ABCD A, B and C are given. Determine the vertex D and shade the visible part of the parallelogram.

   a) A(4/0/2), B(10/9/5), C(10/6/8) (↓6, →6, ↕11)
   b) A(10/6/-2), B(4/12/1), C(-1/7/6) (↓7, →5, ↕14)

☞ *trace lines: lines of intersection of the plane with the coordinate planes.*

7. In a plane A, B and C are given. Construct the trace lines of the planes and thus show the visible parts.

   a) A(2/6/1), B(10/-3/3), C(0/3/4) (↓7, →8, ↕13)
   b) A(3/4/3), B(10/2/7), C(-1/8/5) (↓6, →7, ↕13)
   c) A(7/3/3), B(1/7/-3), C(3/5/3) (↓6, →6, ↕11)
   d) A(10/-2/4), B(-1/5/4), C(1/9/4) (↓6, →7, ↕10)
   e) A(10/4/1), B(2/8/5), C(4/-3/4) (↓8, →6, ↕14)
   f) A(3/-3/-1), B(-1/5/5), C(6/4/1) (↓7, →6, ↕12)
   g) A(-5/7/2), B(7/3/2), C(3/-1/6) (↓7, →7, ↕14)
   h) A(5/7/4), B(8/2/3), C(4/0/0) (↓5, →5, ↕9)
   i) A(5/7/1), B(5/-1/2), C(4/4/2) (↓7, →4, ↕11)

8. The two planes ABC und DEF intersect. With the aid of the trace lines and the lines of intersection, determine the visible parts.

   a) A(12/0/0), B(0/6/0), C(0/0/3);
      D(5/2/-3), E(-2/3/2), F(8/-2/2) (↓7, →6, ↕13)
   b) A(4/3/2), B(0/3/4), C(4/9/-2);
      D(3/-4/-2), E(-1/7/1), F(2/4/4) (↓7, →6, ↕13)
   c) A(4/4/1), B(-2/7/-2), C(8/2/6);
      D(3/6/-3), E(-2/2/3), F(10/-2/9) (↓8, →7, ↕14)
   d) A(7/4/3), B(1/-2/3), C(4/8/3);
      D(4/4/2), E(12/0/6), F(4/1/3) (↓6, →6, ↕12)

9. The line PQ intersects the plane ABC. Find the point of intersection and shade the visibility of the line and the plane.

a)	A(12/0/0),	B(0/8/0),	C(0/0/6);	
	P(6/-2/2),	Q(0/6/4)		(↓7, →6, ↕12)
b)	A(-2/6/1),	B(5/-2/5),	C(7/2/3);	
	P(6/2/4),	Q(9/5/1)		(↓6, →7, ↕12)
c)	A(4/-2/3),	B(7/4/1),	C(-1/2/4);	
	P(-2/4/-3),	Q(4/1/6)		(↓7, →6, ↕12)
d)	A(6/1/1),	B(4/3/6),	C(-4/5/2);	
	P(10/3/6),	Q(6/4/4)		(↓6, →5, ↕11)
e)	A(-3/6/3),	B(7/-2/3),	C(10/5/3);	
	P(8/-4/6),	Q(4/4/4)		(↓8, →8, ↕12)

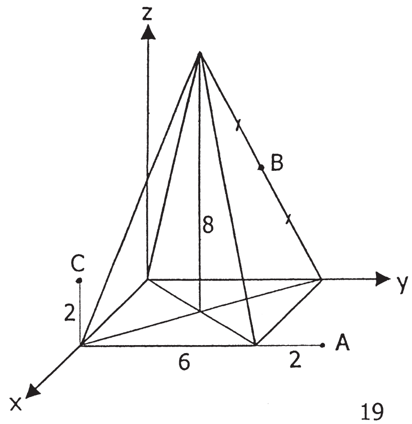
10. Three edges of a cube with side length 6 lie on the positive coordinate axes. It is intersected by a plane ABC. Determine the cross - section and colour the visible parts of the cube and the plane.

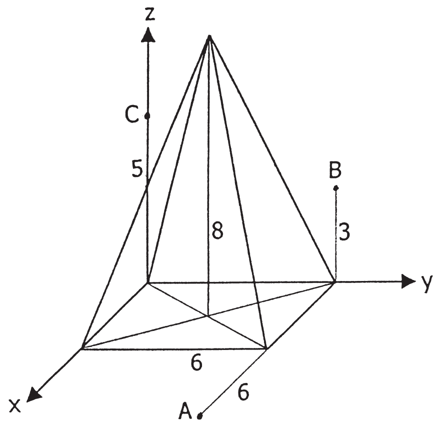
a)	A(10/0/0),	B(0/8/0),	C(0/0/9)	(↓10, →5, ↕15)
b)	A(8/5/2),	B(-4/5/8),	C(8/-5/-2)	(↓11, →9, ↕17)
c)	A(2/5/4),	B(8/11/0),	C(8/2/8)	(↓11, →5, ↕15)
d)	A(3/4/4),	B(6/8/0),	C(3/6.5/2)	(↓9, →5, ↕13)

11. The upper part of a right square - based pyramid (edge of the base 6, height 8) is cut off by the plane through the points A, B and C. Shade the remaining part of the pyramid (☞ below).

a) B is the midpoint of the edge; (↓11, →5, ↕16)

b) (↓8, →8, ↕15)





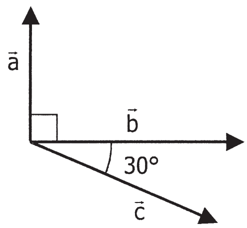
3. BASIC VECTOR OPERATIONS

1. Draw any three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ and illustrate with a construction:
   a) $\vec{a}+\vec{b}=\vec{b}+\vec{a}$ *(commutative law of addition)*
   b) $(\vec{a}+\vec{b})+\vec{c}=\vec{a}+(\vec{b}+\vec{c})$ *(associative law of addition)*

2. Draw any two vectors $\vec{a}$ and $\vec{b}$ and show the following:
   $2\vec{a}+2\vec{b}=2(\vec{a}+\vec{b})$

3. Draw the vectors $\vec{a}$ and $\vec{b}$ with $|\vec{a}| = a = 2$ cm and $|\vec{b}| = b = 3$ cm,
   The angle between $\vec{a}$ and $\vec{b}$ is 40°.
   Construct the vector
   a) $\vec{c}=2\vec{a}-\vec{b}$, b) $\vec{d}=\vec{a}+2\vec{b}$, c) $\vec{e}$, so that $5\vec{a}-2\vec{e}=\vec{b}$.

4. The vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are given (☞ right),
   where : a = 2 cm, b = 2.5 cm, c = 3 cm.
   Construct the vector
   a) $\vec{u}=2\vec{a}+\vec{b}-3\vec{c}$,
   b) $\vec{v}$ given that $3\vec{a}-2\vec{b}+3\vec{c}+\vec{v}=\vec{0}$,
   c) $\vec{w}$ given that $\vec{a}+3\vec{b}-4\vec{c}+2\vec{w}=\vec{0}$.

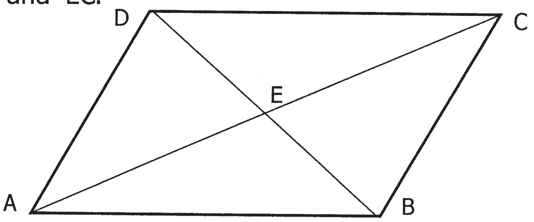


5. Simplify the following equations to express $\vec{a}$ in terms of $\vec{b}$ and $\vec{c}$.
   a) $3\vec{a}-2\vec{b}+\vec{c} = \frac{1}{2}(\vec{a}+4\vec{b})-3\vec{c}$
   b) $\frac{1}{2}(2\vec{a}+\vec{c}) - \frac{3}{4}(3\vec{b}-\vec{a}) = \frac{1}{4}(4\vec{b}+\vec{c})+\vec{a}$
   c) $2(\vec{a}-\vec{b}) - 3(2\vec{b}-5\vec{c}) = \frac{1}{2}(\vec{a}-2\vec{c})$

6. A triangle ABC is given by $\overrightarrow{AB}=\vec{c}$ and $\overrightarrow{BC}=\vec{a}$.
   Point D is the midpoint of the side $\overline{AB}$.
   Express the vectors $\overrightarrow{AC}$, $\overrightarrow{AD}$ and $\overrightarrow{CD}$ in terms of $\vec{a}$ and $\vec{c}$.

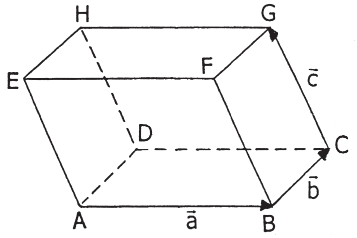
7. In the parallelogram ABCD consider the vectors $\overrightarrow{AB}$, $\overrightarrow{AC}$, $\overrightarrow{AD}$, $\overrightarrow{BC}$, $\overrightarrow{BE}$ and $\overrightarrow{EC}$.
   Let
   a) $\overrightarrow{AB}=\vec{a}$ and $\overrightarrow{BC}=\vec{b}$,
   b) $\overrightarrow{AC}=\vec{e}$ and $\overrightarrow{BD}=\vec{f}$.

   Express the remaining vectors in terms of the given ones.

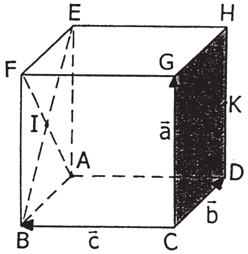


8. In a parallelogram ABCD, let $\overrightarrow{AB} = \vec{a}$ and $\overrightarrow{BC} = \vec{b}$. Point E is the midpoint of $\overline{AB}$; point F lies on $\overline{BC}$ so that $\overline{BF} : \overline{FC} = 3 : 2$ holds. Express the vectors $\overrightarrow{AE}$, $\overrightarrow{AC}$, $\overrightarrow{BD}$, $\overrightarrow{CD}$, $\overrightarrow{DE}$, $\overrightarrow{BF}$, $\overrightarrow{AF}$ and $\overrightarrow{EF}$ in terms of $\vec{a}$ and $\vec{b}$.

9. The parallelepiped shown(☞ right: a solid, made up of 6 parallelograms) is formed by vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$. Express the vectors $\overrightarrow{AC}$, $\overrightarrow{BG}$, $\overrightarrow{AF}$, $\overrightarrow{EC}$, $\overrightarrow{AG}$ and $\overrightarrow{HF}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.



10. A cube (☞ right) is given by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$; K is the midpoint of the edge. Express the vectors $\overrightarrow{CE}$, $\overrightarrow{FD}$, $\overrightarrow{CI}$, $\overrightarrow{BK}$ and $\overrightarrow{IK}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.



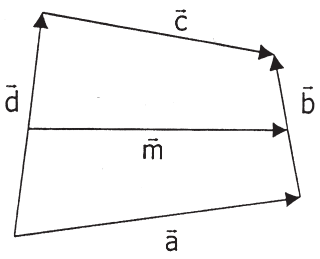
11. In a plane quadrilateral ABCD, let $\overrightarrow{AB} = \vec{a}$, $\overrightarrow{BC} = \vec{b}$ and $\overrightarrow{CD} = \vec{c}$.
   a) Express $\vec{d} = \overrightarrow{DA}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.
   b) Use vectors to show that the midpoints of the sides of the quadrilateral are vertices of a parallelogram.

12. Using a vector method, show that the midline of a triangle is parallel to the base and half as long.

13. Let S be the centroid of a triangle ABC. Prove:
$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = \vec{0}$.

14. Prove: A quadrilateral whose diagonals bisect each other is a parallelogram.

15. The vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$ define a quadrilateral. Vector $\vec{m}$ connects the midpoints of $\vec{b}$ and $\vec{d}$.
Prove: $\vec{m} = ½(\vec{a} + \vec{c})$.



4. VECTORS IN THE COORDINATE SYSTEM

1. Point P(5/2/1) is reflected
   a) in the xy-plane, b) in the xz-plane, c) on the x-axis,
   d) on the z-axis, e) at the origin f) at point S(4/4/4).
   Find the coordinates of the reflected point P'.

2. Describe the locus of the following points:
   a) P(x/0/0) b) P(x/y/0) c) P(x/0/z) d) P(0/y/4)
   e) P(3/y/z) f) P(x/2/z) g) P(x/1/1) h) P(0/a/a)

3. Given the vectors $\vec{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 6 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$,

   a) construct the vector $\vec{d} = 2\vec{a} - \vec{b} - 3\vec{c}$.
   b) Use a calculation to verify your construction.

4. Given the vectors $\vec{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$ and $\vec{c} = \begin{pmatrix} -4 \\ -8 \end{pmatrix}$,

   a) construct the vector $\vec{d} = 3\vec{a} + 2\vec{b} - \vec{c}$.
   b) Use a calculation to verify your construction..

5. Given the vectors $\vec{a} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -12 \\ -6 \\ 0 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} -5 \\ 3 \\ -3 \end{pmatrix}$,

   calculate
   a) $\vec{u} = 2\vec{a} - 1.5\vec{b} + 3\vec{c}$,
   b) $\vec{v} = 3(\vec{a} - 4\vec{b}) - 5\vec{c}$,
   c) $\vec{w} = 5\vec{a} - 2(\vec{b} + 3\vec{c})$.

6. Determine whether vectors $\vec{a}$ and $\vec{b}$ are collinear.

   a) $\vec{a} = \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -12 \\ 3 \\ 9 \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} 5 \\ 7 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2.5 \\ -3.5 \\ 1 \end{pmatrix}$

   c) $\vec{a} = \begin{pmatrix} 8 \\ -16 \\ 28 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -18 \\ 36 \\ -63 \end{pmatrix}$ d) $\vec{a} = \begin{pmatrix} 3 \\ -1 \\ -0.1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2.25 \\ 0.75 \\ 0 \end{pmatrix}$

7. Find the missing values so that each pair of vectors is collinear.

a) $\vec{a} = \begin{pmatrix} -3 \\ 1 \\ 8 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} x \\ -4 \\ z \end{pmatrix}$
b) $\vec{a} = \begin{pmatrix} 6 \\ 0 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -4 \\ y \\ z \end{pmatrix}$

c) $\vec{a} = \begin{pmatrix} x \\ 7 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ y \\ 8 \end{pmatrix}$
d) $\vec{a} = \begin{pmatrix} 4 \\ y \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} x \\ -9 \\ 1 \end{pmatrix}$

8. Let $\vec{a} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -4 \\ 5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

a) The vectors $\vec{d} = 2\vec{a} - \vec{b} + ½\vec{c}$ and $\vec{e} = \begin{pmatrix} x \\ 2 \end{pmatrix}$ are collinear. Find x.

b) The vectors $\vec{f} = 3\vec{a} + 4\vec{b} - 2\vec{c}$ and $\vec{g} = \begin{pmatrix} -4 \\ y \end{pmatrix}$ are collinear. Find y.

9. Let $\vec{a} = \begin{pmatrix} 0 \\ -2 \\ 8 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ -5 \\ -3 \end{pmatrix}$, $\vec{w} = \begin{pmatrix} x \\ y \\ 2.5 \end{pmatrix}$.

The vectors $\vec{v} = \vec{a} - 2\vec{b} + 3\vec{c}$ and $\vec{w}$ are collinear. Find x and y.

10. Determine whether point C lies on the line passing through A and B.
    a) A(-2/5/-4), B(10/-1/0); C(-8/8/-6)
    b) A(6/-3/4), B(2/7/-5); C(-4/22/-18)

11. Determine whether A(1/0/-2), B(3/6/4) and C(-2/-9/-11) are vertices of a triangle ABC.

12. Complete the parallelogram ABCD where A, B and C are given. What are the coordinates of D ?
    a) A(2/-3), B(8/1), C(4/5)
    b) A(-2/5), B(7/-6), C(4/7)
    c) A(2/-1/7), B(5/3/0), C(-1/8/-6)
    d) A(4/0/0), B(-7/0/6), C(3/8/-2)

13. The points A(-1/4/0), B(3/2/5) and C(2/-3/-7) are vertices of a parallelogram. Determine the coordinates of the fourth vertex D (☞ three solutions !).

14. Are A(5/-1/6), B(8/10/-3), C(10/13/-1) and D(7/2/8) vertices of a parallelogram ?

15. Given the vertices A and B and the point of intersection of the diagonals E in a parallelogram ABCD. Find the coordinates of C and D.

a) A(4/-3/-5), B(-2/5/-1), E(-1/2/3) b) A(2/5/-3), B(-1/-2/4), E(3/-4/2)

16. Express vector $\vec{c}$ in terms of vectors $\vec{a}$ and $\vec{b}$. You can check your calculation with a drawing.

a) $\vec{a} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$; $\vec{c} = \begin{pmatrix} 9 \\ 8 \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} 6 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 5 \\ 9 \end{pmatrix}$

c) $\vec{a} = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$; $\vec{c} = \begin{pmatrix} 2 \\ 7 \end{pmatrix}$ d) $\vec{a} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ -1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 3 \\ -3 \end{pmatrix}$

17. Express vector $\vec{d}$ in terms of $\vec{a}$, $\vec{b}$ and $\vec{c}$.

a) $\vec{a} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} -8 \\ -13 \\ 12 \end{pmatrix}$

b) $\vec{a} = \begin{pmatrix} 4 \\ -2 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -2 \\ 3 \\ 7 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 1 \\ 1 \\ -4 \end{pmatrix}$; $\vec{d} = \begin{pmatrix} 2 \\ -12 \\ -20 \end{pmatrix}$

c) $\vec{a} = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ 0 \\ 4 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 5 \\ -6 \\ -7 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} -8 \\ 9 \\ 2 \end{pmatrix}$

d) $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ 3 \\ -4 \end{pmatrix}$; $\vec{d} = \begin{pmatrix} -5 \\ -8 \\ 18 \end{pmatrix}$

e) $\vec{a} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 5 \\ 7 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 0 \\ 5 \\ -4 \end{pmatrix}$; $\vec{d} = \begin{pmatrix} -6 \\ 2 \\ 5 \end{pmatrix}$

f) $\vec{a} = \begin{pmatrix} 3 \\ 0 \\ -4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix}$; $\vec{d} = \begin{pmatrix} 0 \\ 12 \\ -8 \end{pmatrix}$

18. Calculate the length of vector $\vec{a} = \begin{pmatrix} -3 \\ 6 \\ 6 \end{pmatrix}$. Find the components of a vector with the same direction and a length of 22.5.

19. Calculate the length of $\vec{a} = \begin{pmatrix} 12 \\ -20 \\ 9 \end{pmatrix}$. Find the components of a vector of opposite direction and length 40.

20. Calculate the perimeter of the triangle ABC.
   a) A(-9/2), B(12/2), C(-3/10)   b) A(11/5), B(5/13), C(-10/-23)
   c) A(6/3/8), B(10/11/0), C(12/3/16)
   d) A(-1/2/-5), B(1/12/6), C(3/6/-3)

21. The length of the segment $\overline{AB}$ with A(7/1/5) and B(6/y/-3) is 9. Find y.

22. Find the points on the z-axis where the distance to A(-6/3/7) is 7.

23. Where does the sphere with the centre M(2/3/-6) and the radius r = 9 intersect the x-axis ?

24. $|\vec{v}| = \left|\begin{pmatrix} x \\ y \\ 3 \end{pmatrix}\right| = 7$. The x-component is 8 more than the y-component.

   Find x and y.

25. $|\vec{v}| = \left|\begin{pmatrix} 6 \\ y \\ z \end{pmatrix}\right| = 9$. The sum of the three components is 9.

   Find y and z.

26. Find the point P on the x-axis which is equidistant from A(-2/1) and B(4/5). (A sketch may help)

27a) Find point P on the y-axis which is equidistant from A(7/0/-4) and B(-3/1/-7).
   b) Find point P on the y-axis which is equidistant from A(3/4/-7) and B(-1/2/1).
   c) Find point P on the z-axis which is equidistant from A(5/2/-3) and B(-3/6/5).

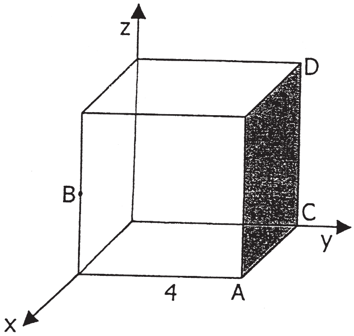
28a) Find the point on the x-axis whose distance to A(0/-2/4) is twice its distance to B(6/2/-1).
   b) Find the point on the y-axis whose distance to A(-5/7/10) is three times its distance to B(3/1/0).
   c) Find the point on the z-axis whose distance to A(11/8/-9) is three times its distance to B(6/-3/5).

29. Find the coordinates of the centroid of the triangle ABC.
a) A(5/3), B(-4/7), C(2/-1)   b) A(6/1/-3), B(7/-7/4), C(-4/0/5)

30. In the triangle ABC, points A, B and the centroid S are given.
Find the coordinates of C as well as the length of the median s_a.
a) A(-3/1/4), B(2/0/5), S(-1/2/2)   b) A(4/-1/3), B(-1/0/5), S(2/1/2)

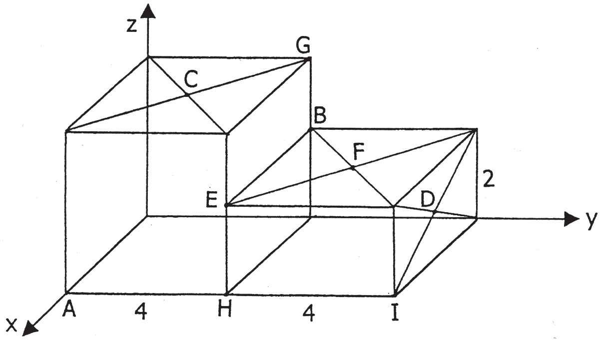
31. B is the midpoint of the edge of the cube.
Find the coordinates of the point

a) on the x-axis which is equidistant from B and C,
b) on the line through C and D, which is equidistant from A and B.



32. A cube (edge 4) and a cuboid (edges 4, 4, 2) are given, as illustrated below.

a) Are the two vectors $\overrightarrow{AG}$ and $\overrightarrow{HF}$ collinear ?
b) The point Q lies in the xy-plane. The vectors $\vec{v} = 2\overrightarrow{AD} - 3\overrightarrow{HD}$ and $\overrightarrow{CQ}$ are collinear. Find the coordinates of Q.
c) Complete the points C, H and D of the parallelogram CHDK. Find the coordinates of K.
d) Which point on the z-axis is equidistant from B and C ?
e) Which point on the edge $\overline{HI}$ of the cuboid has the distance d = 6 to the point G ?



33. In the square ABCD, vertices B(5/0/10) and D(2/21/10) are given; furthermore, vertex A is known to lie in the xy - plane.
Calculate the coordinates of A and C.

5. THE SCALAR PRODUCT

1. Compute the scalar product $\vec{a} \cdot \vec{b}$, given the following:
   a) $a = 4$, $b = 7$, $\measuredangle(\vec{a}, \vec{b}) = 60°$,
   b) $a = 1$, $b = 6$, $\measuredangle(\vec{a}, \vec{b}) = 120°$,
   c) $a = \sqrt{2}$, $b = 12$, $\measuredangle(\vec{a}, \vec{b}) = 45°$,
   d) $a = 6$, $b = 11$, $\measuredangle(\vec{a}, \vec{b}) = 90°$.

2. Compute the angle between $\vec{a}$ and $\vec{b}$, given the following:
   a) $\vec{a} \cdot \vec{b} = 10$, $a = 5$, $b = 4$; b) $\vec{a} \cdot \vec{b} = -3$, $a = 7$, $b = 6$;
   c) $\vec{a} \cdot \vec{b} = 0$, $a = \sqrt{2}$, $b = 9$; d) $\vec{a} \cdot \vec{b} = -40$, $a = 8$, $b = 5$;
   e) $\vec{a} \cdot \vec{b} = -11$, $a = 11$, $b = 2$; f) $\vec{a} \cdot \vec{b} = 36$, $a = 12$, $b = 8$.

3. Prove: a) $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$ *(commutative law)*
   b) $k(\vec{a} \cdot \vec{b}) = (k\vec{a}) \cdot \vec{b} = \vec{a} \cdot (k\vec{b})$
   c) $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$ *(distributive law)*

4. Show that the following relationships apply for all vectors $\vec{a}$ and $\vec{b}$.
   a) $\vec{a} \cdot \vec{b} \leq ab$ b) $(\vec{a})^2 = \vec{a} \cdot \vec{a} = a^2$
   c) $\vec{a} \perp \vec{b}$, if $a \neq 0$ and $b \neq 0$ and $\vec{a} \cdot \vec{b} = 0$

5. Find the angle between $\vec{a}$ and $\vec{b}$ where the following is satisfied:
   a) $\vec{a} \cdot \vec{b} = ab$, b) $\vec{a} \cdot \vec{b} = -ab$, c) $\vec{a} \cdot \vec{b} = \frac{1}{2} ab$, d) $\vec{a} \cdot \vec{b} = -\frac{1}{2} ab$

6. Prove with the aid of the scalar product
   a) Pythagoras' theorem,
   b) the theorem: The diagonals of a rhombus are perpendicular.

7. Find the angle between $\vec{a}$ and $\vec{b}$ when:
   a) $a = \frac{1}{2} b$, $\vec{a} \cdot (\vec{a} - \vec{b}) = 0$, $(a \neq 0)$;
   b) $a = 6$, $b = 5$, $\vec{a} \cdot (\vec{a} - 2\vec{b}) = 6$;
   c) $a = 2$, $b = 7$, $\vec{a} \cdot (3\vec{a} - \vec{b}) = 5$;
   d) $a = 2b$, $(\vec{a} + \vec{b}) \cdot (\vec{a} - 3\vec{b}) = 0$, $(a \neq 0)$;
   e) $a = 3b$, $(\vec{a} - \vec{b}) \cdot (\vec{a} + 4\vec{b}) = 0$, $(a \neq 0)$;
   f) $(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b}) = 0$, $\vec{a} \cdot (\vec{a} + 2\vec{b}) = 0$, $(a \neq 0, b \neq 0)$;
   g) $b = 2a$, $\vec{b} \cdot (3\vec{a} + \vec{b}) = 0$, $(b \neq 0)$;
   h) $a = 6$, $b = 4$, $(2\vec{a} + 3\vec{b}) \cdot (\vec{a} - 4\vec{b}) = -60$;
   i) $(\vec{a} - 2\vec{b}) \cdot (\vec{a} + 2\vec{b}) = 0$, $(\vec{a} - \vec{b}) \cdot (\vec{a} + 3\vec{b}) = 0$, $(a \neq 0, b \neq 0)$.

8. Calculate the scalar product $\vec{a} \cdot \vec{b}$.

a) $\vec{a} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} 5 \\ -8 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 7 \end{pmatrix}$

c) $\vec{a} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 4 \\ -5 \end{pmatrix}$ d) $\vec{a} = \begin{pmatrix} 9 \\ -7 \\ -4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 6 \\ 0 \\ 14 \end{pmatrix}$

9. Given: $\vec{a} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -3 \\ 5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 2 \\ -7 \end{pmatrix}$, $\vec{d} = \begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

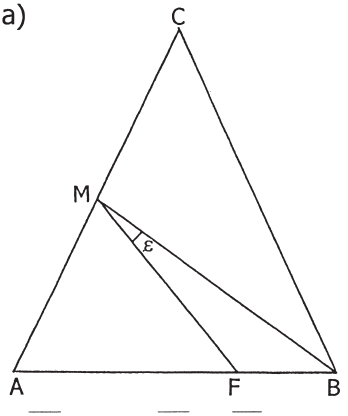
Calculate a) $\vec{a} \cdot \vec{b}$, b) $\vec{a} \cdot (\vec{b} + \vec{c})$, c) $(\vec{a} - \vec{b}) \cdot (\vec{c} + \vec{d})$,
d) $(2\vec{a} + \vec{b}) \cdot (\vec{c} - 3\vec{d})$, e) $(3\vec{b} - \vec{c}) \cdot (3\vec{d} - 4\vec{a})$.

10. Find the angle between vectors $\vec{a}$ and $\vec{b}$.
Use a sketch to verify your calculation.

a) $\vec{a} = \begin{pmatrix} 12 \\ -5 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

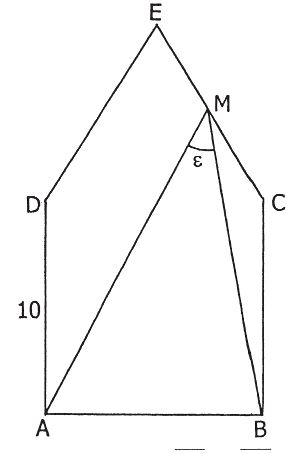
11. The angle ε can be calculated in various ways; here, do it with the scalar product.

a)



$\overline{AB} = 18$, $\overline{AC} = \overline{BC} = 41$,
$\overline{BF} = 6$; $\overline{AM} = \overline{CM}$.

b)



Square ABCD; $\overline{CE} = \overline{DE} = 13$;
$\overline{CM} = \overline{EM}$.

12. Find the angle between $\vec{a}$ and $\vec{b}$.

a) $\vec{a} = \begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 7 \\ -4 \\ 4 \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} -8 \\ 1 \\ 4 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ -12 \\ 5 \end{pmatrix}$

13. Calculate the angles of the triangle ABC.
a) A(-4/2/6), B(-3/6/0), C(0/-2/-1) b) A(4/-1/2), B(1/0/7), C(-3/-2/5)

14. Calculate the angle between $\vec{a}$ and the coordinate axes.

a) $\vec{a} = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix}$ c) $\vec{a} = \begin{pmatrix} -7 \\ 0 \\ 5 \end{pmatrix}$

15. Find z given that the angle between the vectors $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ z \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ z \end{pmatrix}$

is 60°.

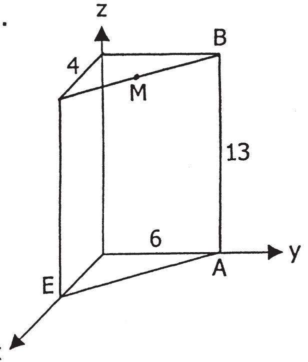
16. Find y given that the angle between the vectors $\vec{a} = \begin{pmatrix} 1 \\ y \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

is 45°.

17. Find the possible values of u given that $\vec{a}$ and $\vec{b}$ are normal (perpendicular).

a) $\vec{a} = \begin{pmatrix} 2 \\ -7 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 5 \\ 3 \\ u \end{pmatrix}$ b) $\vec{a} = \begin{pmatrix} u \\ -1 \\ u-7 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} u+2 \\ 9 \\ u+1 \end{pmatrix}$

18. Determine the point P given that ∡APB = 90°.
a) A(4/2/8), B(6/5/-3), P on the y-axis
b) A(3/-2/1), B(-1/4/3), P on the x-axis
c) A(3/-4/6), B(8/8/8), P on the z-axis
d) A(1/-2/4), B(2/5/1), P on the x-axis



19. M is the midpoint of the edge of the prism. A point P lies on the edge $\overline{AB}$, so that $\overline{EP}$ is the hypotenuse of the triangle EPM. Determine the coordinates of P, as well as the length of the hypotenuse $\overline{EP}$.

20. The vectors $\vec{a} = \begin{pmatrix} x \\ -2 \\ 6 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 2 \\ 6 \\ z \end{pmatrix}$ form the sides of a square.

Calculate the area of this square.

6. THE STRAIGHT LINE

a) Lines in the xy - plane

1. For each of the following, sketch the line and determine its Cartesian equation. The line passes
   a) through A(4/7) and its gradient m is equal to 3;
   b) through A(-2/1) and B(5/3);
   c) through A(5/-2) and intersects the y - axis at y = 4;
   d) through A(-3/-7) and is parallel to the y - axis;
   e) through A(8/-5) and is parallel to the x - axis.

2. Are the lines ℓ_1 and ℓ_2 parallel ?
   a) ℓ_1: $3x - 4y + 5 = 0$, ℓ_2: $y = \frac{3}{4}x - 4$
   b) ℓ_1: $3x - 5y + 4 = 0$, ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \end{pmatrix} + t\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

3. Are P(3/4) and Q(21/15) on the line which is defined by A(-3/0) and the gradient $m = {}^2/_3$?

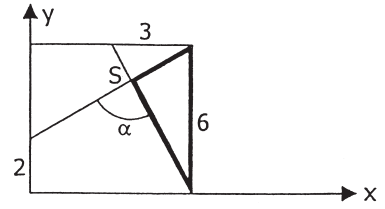
4. Determine the Cartesian equation of the line which passes through P(4/-5) and is parallel to the line
   a) ℓ: $5x - 2y + 4 = 0$, b) ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

5. Given two parallel lines ℓ_1 and ℓ_2. Find the Cartesian equation of the line which lies halfway between them.
   a) ℓ_1: $3x - 2y + 4 = 0$, ℓ_2: $6x - 4y - 3 = 0$
   b) ℓ_1: $2x - 3y + 12 = 0$, ℓ_2: $y = {}^2/_3\, x - {}^1/_2$
   c) ℓ_1: $y = {}^1/_3\, x - 4$, ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

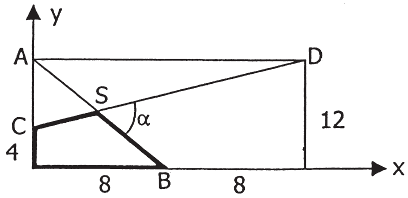
6. Calculate the point and angle of intersection of the lines ℓ_1 and ℓ_2.
   a) ℓ_1: $y = 4x - 7$, ℓ_2: $y = 7x - 4$
   b) ℓ_1: $y = -x - 3$, ℓ_2: $y = 2x + 6$
   c) ℓ_1: $2x - 3y - 3 = 0$, ℓ_2: $3x + 2y - 24 = 0$
   d) ℓ_1: $5x + 2y + 36 = 0$, ℓ_2: $5x - 2y + 4 = 0$
   e) ℓ_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \end{pmatrix} + s\begin{pmatrix} 3 \\ 5 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \end{pmatrix} + t\begin{pmatrix} -1 \\ 1 \end{pmatrix}$
   f) ℓ_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 8 \end{pmatrix} + s\begin{pmatrix} 3 \\ -2 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} + t\begin{pmatrix} -6 \\ 4 \end{pmatrix}$

7. The line ℓ_2 intersects the line ℓ_1: 3x - y + 7 = 0 at an angle of 45°. Calculate the gradient of line ℓ_2.

8. In the given square, determine (☞ right)
   a) the point of intersection S,
   b) the angle α,
   c) the area of the marked triangle.



9. In the given rectangle, determine (☞ right)
   a) the point of intersection S,
   b) the angle α,
   c) the area of the marked quadrilateral.



10. Show that the lines ℓ_1 and ℓ_2 are normal.
   a) ℓ_1: 2x - 5y - 5 = 0 , ℓ_2: 5x + 2y + 10 = 0
   b) ℓ_1: 3x + y - 1 = 0 , ℓ_2: 2x - 6y + 3 = 0
   c) ℓ_1: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + s\begin{pmatrix} 5 \\ 12 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + t\begin{pmatrix} -12 \\ 5 \end{pmatrix}$

11. Find the coordinates of the orthocentre H of the triangle ABC.
   a) A(0/0), B(12/6), C(4/12) b) A(-5/-1), B(11/-9), C(5/9)

12. Reflect the point P in the line ℓ and calculate the coordinates of the reflected point P'.
   a) ℓ: y = x + 3 , P(2/3)
   b) ℓ: 3x - 5y - 5 = 0 , P(2/7)
   c) ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, P(12/-3)
   d) ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -7 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ -3 \end{pmatrix}$, P(0/-4)

13. Reflect the line ℓ_1 in the line ℓ_2 and calculate the Cartesian equation of the reflected line ℓ_1'.
   a) ℓ_1: x + 3y - 21 = 0 , ℓ_2: x - 2y + 4 = 0
   b) ℓ_1: 2x + y - 12 = 0 , ℓ_2: 2x + 3y - 12 = 0
   c) ℓ_1: 15x - 5y - 1 = 0 , ℓ_2: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 \\ 3 \end{pmatrix}$

b) Lines in space

14. Find a vector equation for the line that passes through
    a) A(-4/0/3) and B(3/2/5);
    b) A(2/-1/5) and intersects the x - axis at x = 5;
    c) A(4/3/-3) and is parallel to the z - axis;
    d) A(7/5/3) and is parallel to the y - axis.

15. Given the points A(-2/1/5) and B(4/2/-5), find a vector equation of the line that passes through the midpoint of the segment $\overline{AB}$ and is parallel to the x - axis.

16. A(4/1/3), B(-2/-4/3) and C(7/4/-1) are vertices of a triangle. Find a vector equation of the line that passes through P(1/0/1) as well as through the centroid of the triangle ABC.

17. Are the points A(5/4/2), B(0/-11/-7), C(7.5/11.5/7) on the line

$$\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \\ -4 \end{pmatrix} + t\begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} ?$$

18. Describe the particular position of the line ℓ.

a) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} + t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ b) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

c) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix}$ d) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix}$

19. Determine the trace points of the line through A and B.
    a) A(3/1/6), B(4/-1/9) b) A(5/-1/6), B(3/-2/9)

20. In a triangle ABC where A(1/10/2), B(-1/3/1) and C(-2/0/6), find the trace points in the xy - plane of the lines passing along the sides of the triangle and show that the three points lie on a line.

21. The segment with end-points A(-4/5/-2) and B(5/-1/4) is divided into three equal parts. Find the coordinates of the dividing points.

22. The line ℓ through R(3/-2/1) is parallel to the line through P(1/4/-2) and Q(3/8/-1). Does S(-2/-12/-1.5) lie on the line ℓ?

23. Are the following lines skew, parallel, coincident or intersecting? If they intersect, find their point of intersection.

a) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix} + t\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

b) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -1 \end{pmatrix} + s\begin{pmatrix} 0.8 \\ 0.2 \\ -1.0 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} + t\begin{pmatrix} -4 \\ -1 \\ 5 \end{pmatrix}$

c) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} + s\begin{pmatrix} -0.6 \\ -1.0 \\ 0.2 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ 2 \end{pmatrix} + t\begin{pmatrix} 3 \\ 5 \\ -1 \end{pmatrix}$

d) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \\ 3 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ 5 \end{pmatrix} + t\begin{pmatrix} 4 \\ 3 \\ 1 \end{pmatrix}$

e) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \\ 0 \end{pmatrix} + s\begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 0 \\ -3 \\ 0.5 \end{pmatrix}$

f) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 0 \end{pmatrix} + s\begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

g) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix} + s\begin{pmatrix} 5 \\ -4 \\ 0 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \\ 2 \end{pmatrix} + t\begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix}$

24. Find the point of intersection S and the acute angle between ℓ_1 and ℓ_2.

a) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ -2 \end{pmatrix} + s\begin{pmatrix} 3 \\ -5 \\ 9 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -16 \\ -6 \\ 9 \end{pmatrix} + t\begin{pmatrix} 8 \\ 3 \\ -1 \end{pmatrix}$

b) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 24 \end{pmatrix} + s\begin{pmatrix} 2 \\ -6 \\ 3 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 14 \\ -4 \\ 3 \end{pmatrix} + t\begin{pmatrix} -7 \\ 6 \\ 6 \end{pmatrix}$

c) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 51 \\ 6 \end{pmatrix} + s\begin{pmatrix} 4 \\ -1 \\ 7 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 49 \\ 4 \\ 7 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \\ -6 \end{pmatrix}$

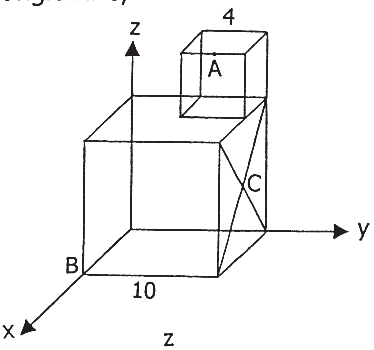
25. The lines ℓ_1 and ℓ_2 intersect at S. Determine the complete vector equation of ℓ_2 and the coordinates of S.

a) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix} + s \begin{pmatrix} 2 \\ 1 \\ -4 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \nabla \\ 5 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ \Theta \\ 4 \end{pmatrix}$, S(✻/10/✿)

b) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + s \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 8 \end{pmatrix} + t \begin{pmatrix} \Diamond \\ \Phi \\ 9 \end{pmatrix}$, S(-4/⊛/⊙)

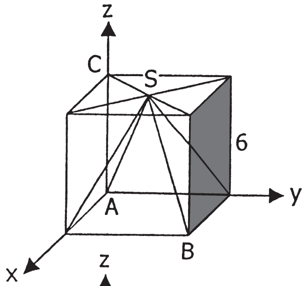
26. In a triangle ABC the vertices A(2/-3/4) and B(7/9/6) are given. The vertex C lies on the line through P(-1/1/4) and Q(-1/1/5).
Calculate the coordinates of the vertex C given that the side c = $\overline{AB}$
a) is the hypotenuse of the right - angled triangle ABC,
b) is the base of the isosceles triangle ABC.

27. Given two cubes (☞ right).
A is the midpoint of an edge.
a) Find vector equations for the lines AB and BC.
b) Calculate the acute angle between these two lines.
c) Where exactly does the line AB intersect the large cube ?



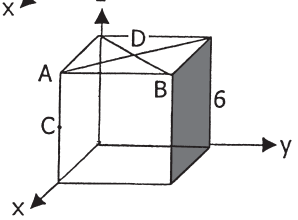
28. A right square - pyramid is inscribed into a cube (☞ right).

Find the point and angle of intersection of the body diagonal $\overline{BC}$ and the edge of the pyramid $\overline{AS}$.



29. Given a cube (☞ right).
C is the midpoint of the edge.

Find a point P on the segment $\overline{CD}$ so that $\measuredangle APB = 90°$.



30. The points A(3/2/1), B(5/3/3) and C(-1/3/9) are given.
Find vector equations for the angular bisectors of the lines AB and AC.

7. THE PLANE

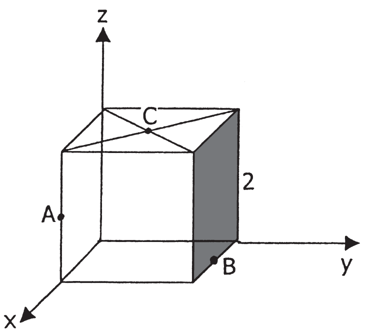
1. Determine the Cartesian equation of the plane ABC.

   a) A(4/1/7), B(3/-1/2), C(2/0/0)
   b) A(4/-2/-2), B(7/2/4), C(0/-5/-3)
   c) A(2/5/4), B(7/0/-3), C(-8/-5/2)
   d) A(0/1/-3), B(7/5/1), C(7/-3/-5)
   e) A(-5/0/6), B(3/-9/1), C(6/-4/0)
   f) A(17/5/-6), B(3/11/9), C(-8/4/14)

2. Determine the Cartesian equation of the plane defined by the point P and the line ℓ.

   a) P(0/-2/0) , ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

   b) P(-2/-6/4) , ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$



3. In the cube on the right, A and B are the midpoints of the two edges. Find the Cartesian equation of the plane ABC.

4. Find the Cartesian equation of the plane defined by the parallel lines p_1 and p_2.

   a) p_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 6 \end{pmatrix} + s \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$, p_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$

   b) p_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ -2 \end{pmatrix} + s \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$, p_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix}$

5. Determine the Cartesian equation of the following planes given by

   a) A(0/5/3) , B(2/-1/-1) , C(-1/6/5);

   b) P(4/-1/1) and g: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -6 \\ -2 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}$;

   c) A(6/0/1) and B(-1/-2/2); also, the z - axis is parallel to the plane.

6. Determine the Cartesian equation of the plane containing

a) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$ and ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix}$;

b) P(4/-2.5/1) , parallel to the xz - plane ;

c) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} + s \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ and ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 8 \end{pmatrix} + t \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}$.

7. Describe the particular position of the following planes.

a) $\mathcal{P}$: $3x - 7y + 3 = 0$
b) $\mathcal{P}$: $2y - 5z + 9 = 0$
c) $\mathcal{P}$: $4x - 5 = 0$
d) $\mathcal{P}$: $2x + 3z = 0$

8. Prove that the lines ℓ_1 and ℓ_2 intersect; determine the Cartesian equation of the plane containing ℓ_1 and ℓ_2.

a) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \\ 1 \end{pmatrix} + s \begin{pmatrix} 5 \\ 7 \\ -1 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

b) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ 8 \end{pmatrix} + s \begin{pmatrix} 3 \\ -1 \\ 4 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ -8 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix}$

9. Calculate the axes intercepts of the plane $\mathcal{P}$.

a) $\mathcal{P}$: $3x - y + 2z - 6 = 0$
b) $\mathcal{P}$: $4x + 9y - 3z + 36 = 0$
c) $\mathcal{P}$: $3x - 5z + 60 = 0$
d) $\mathcal{P}$: $7y - 4z - 4 = 0$

10. Calculate the axes intercepts and sketch an oblique parallel projection of the plane $\mathcal{P}$ with its visible trace lines.

a) $\mathcal{P}$: $4x + 2y + 6z - 24 = 0$
b) $\mathcal{P}$: $2x + 5y - 4z - 20 = 0$
c) $\mathcal{P}$: $2x + 4y - 3z - 18 = 0$
d) $\mathcal{P}$: $5x - 6y + 30 = 0$

11. Determine the Cartesian equation for the plane with axes intercepts x = a , y = b and z = c. Then divide the equation by abc.

12. A plane is given by its axes intercepts. Determine its Cartesian equation.

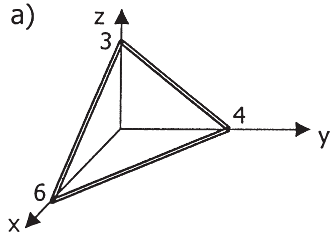
a) a = 2 , b = 3 , c = 3
b) a = 8 , b = -6 , c = 1
c) a = -9 , b = 5 , c = 3
d) a = b = -14 , c = 6
e) a = -5 , b = -7
f) a = 7

13. Determine b and c in the equation of the plane $\mathscr{P}$: $3x + by + cz - 24 = 0$ given the following:
   a) the x - intercept value is twice the y - intercept value; the sum of all three intercepts values equals 0;
   b) the z - intercept value is 3 greater than the y-intercept value. The product of these two values equals 40.

14. Determine vector equations of the trace lines of plane $\mathscr{P}$.
   a) $\mathscr{P}$: $x - 3y + z - 6 = 0$
   b) $\mathscr{P}$: $4y - z + 8 = 0$

15. Determine the Cartesian equation for each illustrated plane.

a)



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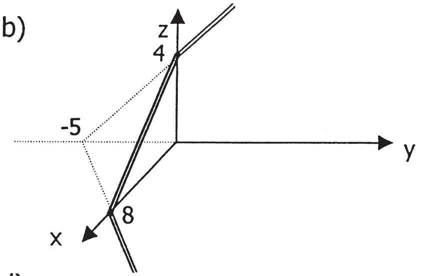
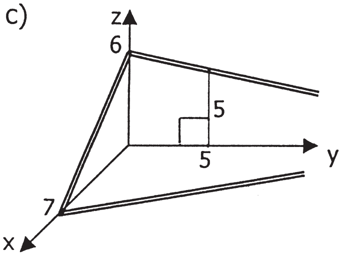


c)



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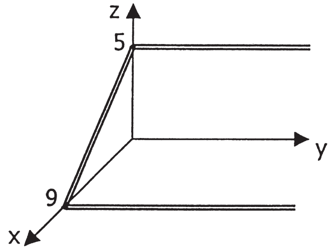

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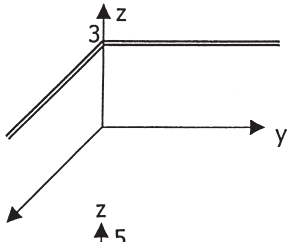

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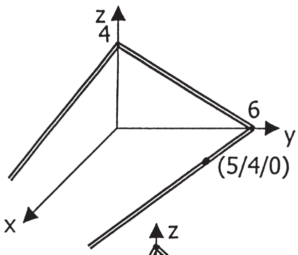

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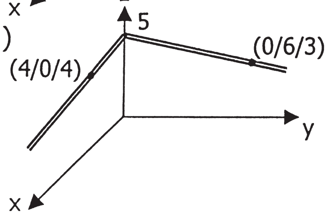

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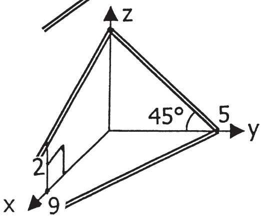


16. Do the points A and B lie in the plane $\mathcal{P}$?

a) $\mathcal{P}$: 3x - 2y - 5z + 11 = 0 ; A(5/-2/6), B(4/6/2)

b) $\mathcal{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix} + u \begin{pmatrix} -1 \\ 2 \\ 2 \end{pmatrix} + v \begin{pmatrix} 4 \\ -4 \\ -3 \end{pmatrix}$; A(2/4/1), B(1/6/3)

17. Find point P in plane $\mathcal{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix} + u \begin{pmatrix} 9 \\ -8 \\ -3 \end{pmatrix} + v \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$

a) which lies on the z - axis,
b) which possesses three equal components,
c) with horizontal projection P'(-3/4/0),
d) with vertical projection P"(0/1/7).

18. Determine whether A(2/1/10), B(0/4/-3), C(-1/0/5) and D(3/5/2) are vertices of a quadrilateral.

19. Find the coordinates of the point of intersection of the three planes $\mathcal{P}_1$: 3x - 2y+z - 8 = 0 , $\mathcal{P}_2$: x+y - 3z+4 = 0 and $\mathcal{P}_3$: 4x+3y - 5z - 8 = 0.

20. The plane $\mathcal{P}$: 9x + 16y + cz - 144 = 0 intersects the coordinate axes at the points A, B and C. O is the origin.
Find c given that the pyramid OABC has the volume V = 384.

21. Find the point where the line ℓ intersects the plane $\mathcal{P}$.

a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ -10 \\ 8 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}$, $\mathcal{P}$: 3x - y + 4z - 15 = 0

b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} + t \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$, $\mathcal{P}$: 5x + 6y - 2z - 20 = 0

c) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 2 \\ 9 \end{pmatrix} + t \begin{pmatrix} -2 \\ 3 \\ 4 \end{pmatrix}$, $\mathcal{P}$: 4y - 3z + 27 = 0

d) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -7 \\ 18 \end{pmatrix} + t \begin{pmatrix} 0 \\ 2 \\ -3 \end{pmatrix}$, $\mathcal{P}$: 2x + 5y - 21 = 0

22. Find the point at which the line ℓ intersects the plane $\mathscr{P}$.

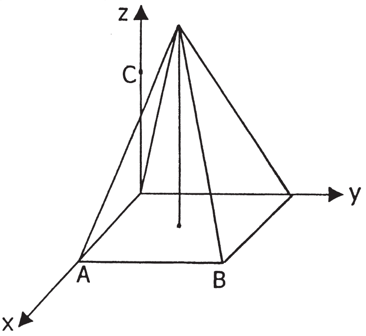
a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -11 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 7 \\ -2 \end{pmatrix}$, $\mathscr{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 7 \end{pmatrix} + u \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} + v \begin{pmatrix} 2 \\ -1 \\ 8 \end{pmatrix}$

b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix} + t \begin{pmatrix} 3 \\ -4 \\ 7 \end{pmatrix}$, $\mathscr{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} + u \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix} + v \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

c) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} + t \begin{pmatrix} 4 \\ 5 \\ -2 \end{pmatrix}$, $\mathscr{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \\ 9 \end{pmatrix} + u \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} + v \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$

d) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ 5 \\ 8 \end{pmatrix} + t \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, $\mathscr{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + u \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} + v \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$

23. Given the point C(0/0/6) as well as a right square - based pyramid with its base edge e = 4 and height = 9 (☞ right).



a) Find the Cartesian equation of the plane ABC.

b) Determine the shape and area of the cross section of the plane and pyramid.

24. Determine a vector equation of the line of intersection between plane $\mathscr{P}_1$ and $\mathscr{P}_2$.

a) $\mathscr{P}_1$: $x - y + 2z - 6 = 0$, $\mathscr{P}_2$: $x + y - 4z + 2 = 0$
b) $\mathscr{P}_1$: $x + 2y - z + 4 = 0$, $\mathscr{P}_2$: $x - 4y + z - 2 = 0$
c) $\mathscr{P}_1$: $2x + y - z + 1 = 0$, $\mathscr{P}_2$ given by A(-1/1/0), B(1/0/2), C(2/0/1)
d) $\mathscr{P}_1$: $x - 2y - z = 0$, $\mathscr{P}_2$ given by A(1/0/1) and the line

ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$

25. Suppose planes $\mathscr{P}_1$: $4x + 3y + 2z + 1 = 0$, $\mathscr{P}_2$: $x + 2y + 3z + 4 = 0$ and $\mathscr{P}_3$: $x + by + cz + 7 = 0$ possess a common line of intersection. Find b and c.

8. NORMAL FORMS

a) The normal vector

1. Determine the Cartesian equation of the plane that passes through point P and is parallel to the plane $\mathcal{P}$.
   a) P(-3/4/2) , $\mathcal{P}$: $2x - y + 6z + 2 = 0$
   b) P(0/5/1) , $\mathcal{P}$: $7x + 2y - 2z + 5 = 0$
   c) P(3/1/3) , $\mathcal{P}$: $3x + z - 2 = 0$

2. Determine the Cartesian equation of the plane that contains the point P and is normal to the line ℓ.
   a) P(-3/3/6) , ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$
   b) P(10/-3/-2) , ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -2 \\ 0 \\ 5 \end{pmatrix}$
   c) P(4/-2/-4) , ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 6 \\ -2 \end{pmatrix}$

3. Find the equation of the plane which is normal to the line segment $\overline{AB}$ and passes through its midpoint.
   a) A(3/1/1) , B(-5/3/7)   b) A(-3/6/8) , B(2/5/4)

4. Calculate the distance between the point P(3/2/1) and the line that passes through the points A(0/9/10) and B(9/9/19).

5. Which point on the line ℓ is equidistant from the points A and B ?
   a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, A(-1/2/1) , B(3/4/-7)
   b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 4 \\ 5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$, A(-3/6/5) , B(5/2/-3)

6. Line ℓ_1 passes through the points A(7/5/1) and B(4/-1/3). Line ℓ_2 passes through the origin and intersects line ℓ_1 at a right angle.
   Find a vector equation of the line ℓ_2.

7. Consider the triangle ABC where A(0/1/5), B(-7/3/-5) and C(5/-3/7), and its altitude $h_a = \overline{AD}$ are given.
   Find the coordinates of D as well as the area of the triangle.

8. Determine the Cartesian equation of the plane which contains the points P and Q and which is normal to the plane $\mathcal{P}$.
   a) P(1/3/0) , Q(3/2/1) , $\mathcal{P}$: 4x + 3y - 2 = 0
   b) P(2/0/1) , Q(-3/-4/2) , $\mathcal{P}$: 3x + y + 7z - 2 = 0
   c) P(3/-6/7) ,Q(3/-3/0) , $\mathcal{P}$: 3y - 2z + 8 = 0

9. Point P is reflected in the plane $\mathcal{P}$. Find the coordinates of the image point P'.
   a) P(0/4/-5) , $\mathcal{P}$: 4x - 3y + z + 4 = 0
   b) P(4/0/-2) , $\mathcal{P}$: x - 2y + 3z - 5 = 0
   c) P(-1/-6/17) , $\mathcal{P}$: 3x - 8z - 7 = 0

10. Point P'(0/0/7) is the reflection point of P(4/3/-2). Find the Cartesian equation for the plane $\mathcal{P}$ in which P was reflected.

11. The line ℓ is reflected in the plane $\mathcal{P}$. Determine a vector equation of the reflected line ℓ'.

   a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix} + t\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$, $\mathcal{P}$: 4x + 2y - z + 1 = 0

   b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 3 \end{pmatrix}$, $\mathcal{P}$: x - 3y - 2z + 42 = 0

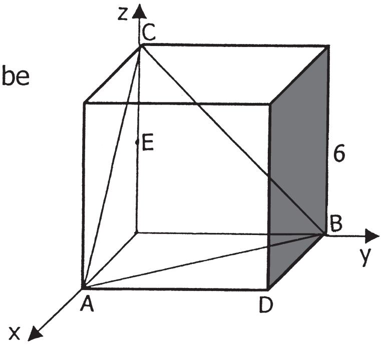
   c) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 7 \\ 3 \end{pmatrix}$, $\mathcal{P}$: 5y - 2z + 9 = 0

   d) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 14 \end{pmatrix} + t\begin{pmatrix} 7 \\ -3 \\ -9 \end{pmatrix}$, $\mathcal{P}$: 3x - 2y + 3z - 1 = 0

12. A ray of light passes through P(7/-7/4) and is reflected in the plane $\mathcal{P}$: 5x - 2y + 3z - 23 = 0. Point Q(7/-1/8) lies on the reflected ray of light.
    At which point in the plane $\mathcal{P}$ is the ray of light reflected ?

13. A ray of light which originates at a light source P(14/7/-11), is reflected in the plane $\mathscr{P}$ at R(5/1/4) and then passes through the point Q(3/13/2). Find the Cartesian equation of the plane $\mathscr{P}$.

14. E is the midpoint of the edge of the glass cube (☞ right). ABC is a plane of reflection. A ray of light starts at D in the direction of point E and is reflected in the plane ABC.
    a) What is the equation of the reflected ray of light?
    b) At which point does this ray of light penetrate the cube?



15. Determine the acute angle between the planes $\mathscr{P}_1$ and $\mathscr{P}_2$.
    a) $\mathscr{P}_1$: $5x - y - 6z + 1 = 0$, $\mathscr{P}_2$: $4x + z - 3 = 0$
    b) $\mathscr{P}_1$: $3x - 2y + 5z - 2 = 0$, $\mathscr{P}_2$ given by P(3/-1/4) and

$$\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix} + t\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$$

    c) $\mathscr{P}_1$: $2x + 4y - 3z + 7 = 0$, $\mathscr{P}_2$ given by the parallel lines

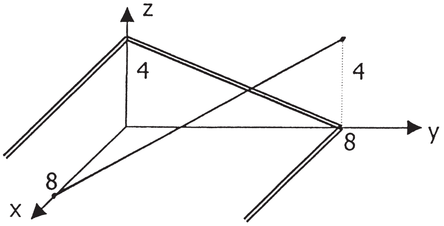
$$p_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -1 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix} \text{ and } p_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ -5 \\ 1 \end{pmatrix}$$

16. Determine the angle of intersection of line ℓ and plane $\mathscr{P}$.

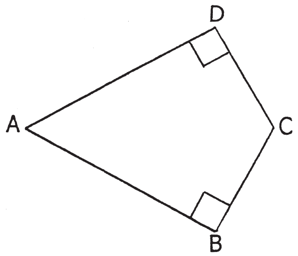
a) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, $\mathscr{P}$: $3x - 4y + 6 = 0$

b) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 3 \\ 1 \\ -1 \end{pmatrix}$, $\mathscr{P}$: $2x - 5y + z + 3 = 0$

17. Find the point and angle of intersection of the line and the plane (☞ right).



18. Consider the right-angled kite ABCD (☞ right) where A(0/-4/0) and C(8/-4/-4) are given. B lies on the x-axis.
    a) Calculate the coordinates of B and D.
    b) Find the size of the angle between the x-axis and the plane of the kite.



19. Find the value of u given that the plane $\mathcal{P}$: $4x + uz - 1 = 0$ and the xy-plane intersect at an angle of 45°.

20. Consider a right circular cone described by the vertex V, the vector $\vec{v}$ which lies along the axis of the cone and the point P which lies on the periphery of the base circle. Find the volume of the following cones.

    a) V(6/-1/7), P(-1/7/5), $\vec{v} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$ b) V(0/1/-4), P(9/1/5), $\vec{v} = \begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}$

21. V(-7/-3/14) is the vertex of a right circular cone. M(3/-1/3) is the centre of the base circle. P(1/-1/8) lies on the curved surface of the cone. Calculate the volume of the cone.

22. In a square ABCD A(3/2/1), B(-3/-1/-5) and C(3/y/z) are given. This square is the base of a right pyramid with the volume 324.
    a) Calculate the *integer* values of y and z.
    b) Find the coordinates of the vertex V (2 solutions !).

23. In a square ABCD A(5/4/-3), B(-2/8/1) and C(2/$y \neq 0$/0) are given.
    a) Determine the y-coordinate of C and the coordinates of D.
    b) The square is rotated through an angle of 90° about side $\overline{AB}$. Find the coordinates of the vertices C' and D' of the rotated square (2 solutions !).

24. The points A(12/10/0), B(9/7/12) and C(-2/2/8) are given.
    a) Show that the triangle ABC is an isosceles right-angled triangle.
    b) Calculate the coordinates of D given that ABCD is a square.
    c) Find the height of the right square-based pyramid ABCDV with the volume 1944 .
    d) Determine the coordinates of the vertex V of this pyramid (2 solutions !).

b) The Hessian normal form

25. What is the distance between point P and the line ℓ ? Check your answer with a sketch.
    a) P(-2/5), ℓ: 3x + 4y - 4 = 0 b) P(3/-1), ℓ: y = 2.4x - 0.4
    c) P(6/5), ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5 \\ -2 \end{pmatrix} + t \begin{pmatrix} 15 \\ 8 \end{pmatrix}$

26. Find the length of the perpendicular to the line ℓ from the origin.
    a) ℓ: 4x - 3y + 20 = 0 b) ℓ: y = $^7/_{24}$ x + 3
    c) ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ d) ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 15 \\ 10 \end{pmatrix} + t \begin{pmatrix} 117 \\ -44 \end{pmatrix}$

27. Calculate the distance between point P and plane $\mathcal{P}$.
    a) P(3/5/1), $\mathcal{P}$: 4x + 7y - 4z + 2 = 0
    b) P(7/-2/-5), $\mathcal{P}$: 6x - 9y - 2z + 7 = 0
    c) P(-4/0/29), $\mathcal{P}$: 15x + 8y - 8 = 0

28. A(0/0/0), B(2/1/0), C(1/2/0), D(1/1/2) are vertices of the pyramid ABCD.
    a) Calculate the height of the pyramid above the base BCD.
    b) Find the angle between the base BCD and the side edge $\overline{AC}$.

29. Given a plane $\mathcal{P}_1$: 4x - 4y - 7z + 6 = 0. Find another plane $\mathcal{P}_2$, passing through the point P(3/-4/1) and parallel to $\mathcal{P}_1$. Determine the Cartesian equation of the plane $\mathcal{P}_2$ as well as the distance between the two planes.

30. Find the distance between the skew lines ℓ_1 and ℓ_2.
    a) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 6 \\ -1 \\ 2 \end{pmatrix}$
    b) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 4 \end{pmatrix} + t \begin{pmatrix} 1 \\ 12 \\ 0 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + t \begin{pmatrix} 0 \\ 12 \\ -1 \end{pmatrix}$
    c) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ -2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 2 \\ -3 \end{pmatrix}$
    d) ℓ_1: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 6 \end{pmatrix}$, ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ 4 \end{pmatrix} + t \begin{pmatrix} 5 \\ 2 \\ 6 \end{pmatrix}$ $\mathcal{P}$

31. Determine the Cartesian equation of the plane that is parallel to the plane $\mathscr{P}$ and d units away from it.
    a) $\mathscr{P}$: $11x - 2y + 10z - 15 = 0$, $d = 3$
    b) $\mathscr{P}$: $24x - 7z + 5 = 0$, $d = 4$
    c) $\mathscr{P}$: $9x + 12y + 8z - 6 = 0$, $d = 2$

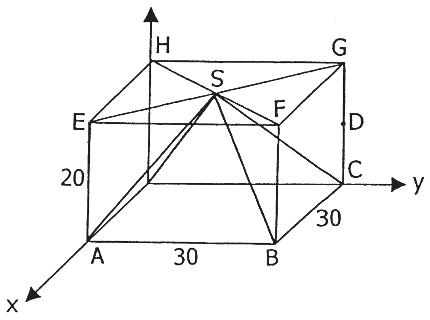
32. Determine the Cartesian equation of the angle bisector planes of the given planes $\mathscr{P}_1$ and $\mathscr{P}_2$.
    a) $\mathscr{P}_1$: $4x - 2y - 4z + 3 = 0$, $\mathscr{P}_2$: $x + 2y - 2z + 5 = 0$
    b) $\mathscr{P}_1$: $6x + 6y + 17z - 2 = 0$, $\mathscr{P}_2$: $15x - 10y - 6z + 9 = 0$
    c) $\mathscr{P}_1$: $10x - 11y + 2z - 11 = 0$, $\mathscr{P}_2$: $4y - 3z - 8 = 0$
    d) $\mathscr{P}_1$: $3x + 6y - 2z - 10 = 0$, $\mathscr{P}_2$: $4x + 8y - z - 10 = 0$

33. Which points on the line ℓ are equidistant from the planes $\mathscr{P}_1$ and $\mathscr{P}_2$?
    a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $\mathscr{P}_1$: $3x - 4y + 2 = 0$, $\mathscr{P}_2$: $4x + 3z + 7 = 0$
    b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} + t \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$, $\mathscr{P}_1$: $14x - 7y - 22z + 38 = 0$,
    $\mathscr{P}_2$: $4x + 7y - 4z + 2 = 0$

34a) What is the distance from point P(3/3/5) to the plane
    $\mathscr{P}$: $x - 12y + 12z + 7 = 0$?
  b) A point Q(x/1/1) is equidistant from plane $\mathscr{P}$ and the xy - plane. Find x.

35a) Find the points on the z - axis that are equidistant from the planes
    $\mathscr{P}_1$: $7x + 4y - 4z - 26 = 0$ and $\mathscr{P}_2$: $x - 4y + 8z + 10 = 0$.
  b) Which of these points is nearer to $\mathscr{P}_1$ and $\mathscr{P}_2$?

36. A pyramid is inscribed into a cuboid (☞ right).
    D is the midpoint of the edge.
    Point P lies on the line segment $\overline{AD}$, which is equidistant from the planes BCS and EFGH.
    a) Find the coordinates of P.
    b) Find the distance.

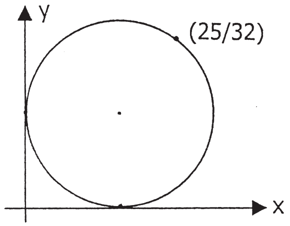


9. CIRCLES AND SPHERES

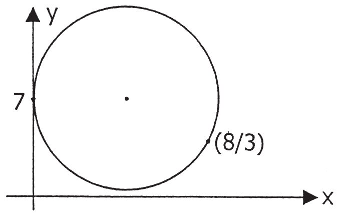
a) The circle in the xy-plane

1. Determine the centre and the radius of the circle.

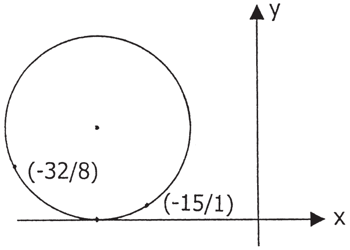
a)



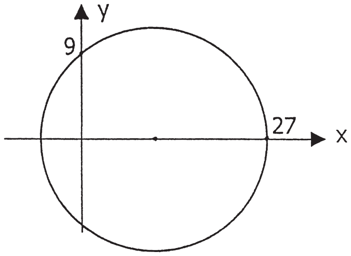
b)



c)



d)



2. Determine the centre O and the radius r of the circle c.
   - a) c: $x^2 + y^2 - 8x + 6y = 0$
   - b) c: $x^2 + y^2 + 2x + 12y + 1 = 0$
   - c) c: $x^2 + y^2 + 14x - 15 = 0$
   - d) c: $x^2 + y^2 - 7x - 8y - 2 = 0$
   - e) c: $2x^2 + 2y^2 - 10x + 6y - 1 = 0$
   - f) c: $3x^2 + 3y^2 + 4x - 18y + 28 = 0$

3. Determine the points of intersection of the circle c and the line ℓ.
   - a) c: $x^2 + y^2 - 100 = 0$, ℓ: $y = 2x + 10$
   - b) c: $x^2 + y^2 + 6x - 8y + 25 = 0$, ℓ: $3x + y + 5 = 0$
   - c) c: $x^2 + y^2 - 4x - 25 = 0$, ℓ: $3x + 7y - 35 = 0$
   - d) c: $x^2 + y^2 - 10x - 4y - 4196 = 0$, ℓ: $3x + 41y - 2632 = 0$

4. Determine the points of intersection of the circle c and the line ℓ as well as the distance from the centre of the circle to the line ℓ.
   - a) c: $(x + 7)^2 + (y + 1)^2 = 50$, ℓ: $3x + 4y = 0$
   - b) c: $x^2 + y^2 - 169 = 0$, ℓ: $12x + 5y - 169 = 0$

5. Point A(-1/6) lies on a circle with a radius r = 5.
   The centre of the circle lies on the line ℓ: $y = -x + 10$.
   Determine the equation of the circle.

6. Point P(x < 0/14) lies on the circle c: $x^2 + y^2 - 12x - 16y = 0$.
Determine the equations of the circles with a radius r = 5 which touch the circle c at point P.

7. Determine the equation of the circle with the centre O which touches the line ℓ.
a) O(2/3) , ℓ: $3x - 4y + 10 = 0$ b) O(3/6) , ℓ: $24x - 7y - 5 = 0$

8. Find the points on circle c: $(x - 1)^2 + (y + 5)^2 = 65$ that are equidistant from the points A(-4/7) and B(2/3).

9. Determine the equations of the circles with radius r = 51 that touch the line ℓ: $15x - 8y - 4 = 0$ at the point P(4/y).

10. Find the equation of the circumcircle of the triangle ABC.
a) A(2/-3) , B(8/3) , C(-2/3) b) A(16/-6) , B(-1/11) , C(-8/4)

11. Find the Cartesian equation of the centreline of the circles
c_1 : $x^2 + y^2 - 6x + 10y - 15 = 0$ and c_2 : $4x^2 + 4y^2 + 12x - 16y + 9 = 0$.

12. A secant ℓ_2 is to be laid through the point P(7/y<0) of the circle
c: $(x - 14)^2 + (y - 9)^2 = 338$, parallel to the line ℓ_1: $12x + 5y - 8 = 0$.
a) Determine the Cartesian equation of the line ℓ_2.
b) Find the length of the chord that the secant ℓ_2 cuts out of the circle.

13. The circle c: $(x - 3)^2 + (y + 2)^2 = 49$ is touched by circles, each with radius r = 3, whose centres lie on the line ℓ: $x - 7y + 33 = 0$.
Find the equations of the circles.

14. The circles c_1 : $x^2 + y^2 = 169$ and c_2 : $(x - 32)^2 + (y - 24)^2 = 1369$ have a common chord. What is its length?

15. A circle whose centre is on the line ℓ_1: $x - 2y - 1 = 0$ touches the lines ℓ_2: $3x + 4y + 22 = 0$ and ℓ_3: $4x - 3y + 46 = 0$.
Find the equation of the circle.

16. A circle with radius r = 10 passes through the point P(-2/7) and touches the line ℓ: $4x + 3y - 63 = 0$. Determine the equation of the circle.

17. A circle passes through A and B and touches ℓ. Find its equation.
a) A(6/10) , B(-11/-7) , ℓ: $x - 14 = 0$
b) A(-3/-4) , B(11/10) , ℓ: $y + 6 = 0$

18. Find the Cartesian equation of the tangent at the point P of the circle c.
   a) c: $x^2 + y^2 = 100$, P(6/8) b) k: $(x - 2)^2 + (y + 3)^2 = 169$, P(-3/9)
   c) c: $x^2 + y^2 - 12x + 20y - 105 = 0$, P(2/5)

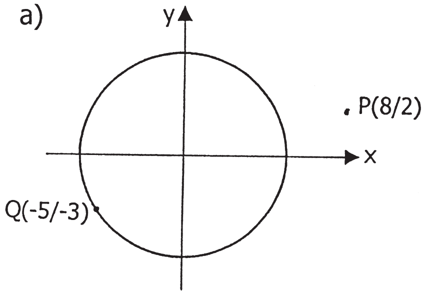
19. Determine the Cartesian equations of the tangents of the circle c which are parallel to the line ℓ.
   a) c: $(x - 1)^2 + (y + 3)^2 = 25$, ℓ: $4x + 3y - 7 = 0$
   b) c: $(x + 2)^2 + y^2 = 625$, ℓ: $7x - 24y + 9 = 0$

20. Determine the Cartesian equations of the tangents of the circle c which are normal to the line ℓ.
   a) c: $(x - 5)^2 + (y - 2)^2 = 100$, ℓ: $3x - 4y + 10 = 0$
   b) c: $x^2 + y^2 + 16y - 225 = 0$, ℓ: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \end{pmatrix} + t\begin{pmatrix} -8 \\ 15 \end{pmatrix}$

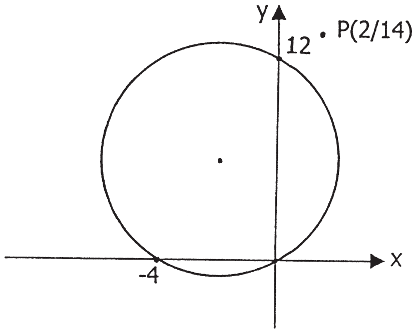
21a) Find the equations for the tangents of the circle c: $x^2 + y^2 = 10$ through the point P(-4/2).
   b) Show that these tangents are normal.

22. Find the Cartesian equations of the tangents to the circle through point P.

a)



b)



b) The sphere

23. Determine the equation of the sphere with diameter $\overline{AB}$.
   a) A(1/-3/0) , B(5/1/4) b) A(3/0/-2) , B(1/6/4)

24. Determine the centre O and the radius r of the sphere S.
   a) S: $x^2 + y^2 + z^2 - 4x + 2y - 10z + 26 = 0$
   b) S: $x^2 + y^2 + z^2 + 12x - 6z + 9 = 0$
   c) S: $x^2 + y^2 + z^2 - 14x + 4y + 53 = 0$
   d) S: $2x^2 + 2y^2 + 2z^2 - 2x + 6y - 4z - 11 = 0$

25. The sphere $\mathcal{S}$: $x^2 + y^2 + z^2 - 4x + 12z - 19 = 0$ is given. Point P(-2/8/13) lies on a sphere which is concentric to $\mathcal{S}$. Find its radius.

26. Find the points of intersection between the sphere $\mathcal{S}$ and the line ℓ passing through A and B.
   a) $\mathcal{S}$: $x^2 + y^2 + z^2 = 41$; A(5/2/1) , B(6/-1/2)
   b) $\mathcal{S}$: $x^2 + y^2 + z^2 = 29$; A(4/-1/1) , B(5/2/0)
   c) $\mathcal{S}$: $(x - 2)^2 + (y + 5)^2 + z^2 = 81$; A(9/5/16) , B(14/7/21)
   d) $\mathcal{S}$: $(x + 1)^2 + (y + 4)^2 + (z - 2)^2 = 49$; A(6/0/0) , B(4/1/0)

27. The spheres $\mathcal{S}_1$: $(x - 1)^2 + (y + 3)^2 + z^2 = 54$ and $\mathcal{S}_2$: $x^2 + y^2 + z^2 - 22x - 4y + 10z + 126 = 0$ are given.
   a) Show that $\mathcal{S}_1$ and $\mathcal{S}_2$ touch each other.
   b) Determine the point of contact.

28. Show that the line ℓ is a tangent to the sphere $\mathcal{S}$ and find the point of contact.
   a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 7 \end{pmatrix} + t \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix}$, $\mathcal{S}$: $(x + 3)^2 + (y - 5)^2 + (z + 4)^2 = 49$
   b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -6 \\ 8 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ -2 \end{pmatrix}$, $\mathcal{S}$: $(x - 4)^2 + y^2 + (z - 5)^2 = 121$

29. Find the equation of the sphere with centre O which touches the plane $\mathcal{P}$.
   a) O(6/5/-3) , $\mathcal{P}$: $x - 2y + 2z + 4 = 0$
   b) O(-4/30/-5) , $\mathcal{P}$: $5x - 14y + 2z = 0$
   c) O(-2/-7/11) , $\mathcal{P}$: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} + u \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} + v \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$

30. Find the equations of the spheres with centre O which touch the given sphere $\mathcal{S}$.
   a) O(9/1/5) , $\mathcal{S}$: $(x - 2)^2 + (y + 3)^2 + (z - 1)^2 = 49$
   b) O(5/4/6) , $\mathcal{S}$: $(x + 1)^2 + (y - 2)^2 + (z - 3)^2 = 25$

31. Given the sphere $\mathcal{S}$: $x^2 + y^2 + z^2 + 6y - 10z - 135 = 0$. Determine the centre and the radius of the trace circle in the xy - plane.
(trace circle: circle of intersection of the sphere and a coordinate plane).

32. The sphere $\mathcal{S}$: $x^2 + y^2 + z^2 - 2y - 22z - 103 = 0$ intersects the plane $\mathcal{P}$: $2x + 2y - z - 18 = 0$ in a circle. Determine the centre and the radius of this circle.

33. Point P(3/1/-2) lies on a sphere with radius r = 3. The centre of the sphere lies on the line that passes through A(1/5/-3) and B(0/7/-2). Find the equation of the sphere.

34. Determine the equations of the spheres with a given radius r which touch the plane ABC at point P.
    a) r = 18 , A(0/-2/4), B(-1/2/-2), C(-10/6/0), P(3/-2/z)
    b) r = 42 , A(6/1/0), B(0/-3/7), C(-3/-3/1), P(10/y/8)

35. Point P lies on the sphere with the centre O and the radius r. Find an equation of the tangent plane to the sphere at point P.
    a) O(1/-3/0) , r = 5 , P(1/1/-3) b) O(3/0/1) , r = 5 , P(3/-4/-2)
    c) O(7/-4/-2) , r = 9 , P(3/3/z>0) d) O(0/1/2) , r = 7 , P(-2/y<0/-4)

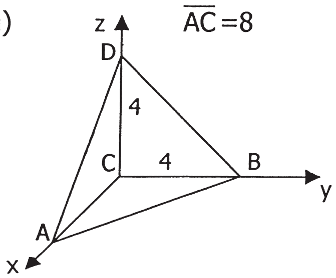
36. Determine Cartesian equations for the tangent planes to the sphere $\mathcal{S}$ which are parallel to plane $\mathcal{P}$.
    a) $\mathcal{S}$: $(x - 3)^2 + (y - 1)^2 + (z + 2)^2 = 49$, $\mathcal{P}$: $3x + 2y - 6z = 0$
    b) $\mathcal{S}$: $x^2 + y^2 + z^2 - 8x + 2z - 64 = 0$, $\mathcal{P}$: $2x - 2y + z - 7 = 0$

37. A ray of light, starting at the light source Q(5/38/-7), travels in the direction of P(3/22/-6) and is reflected in the sphere $\mathcal{S}$: $(x - 3)^2 + (y + 8)^2 + z^2 = 225$.
    a) Find the point R on the sphere where the reflection takes place.
    b) Determine a vector equation of the reflected ray of light.
    c) Find the angle between the rays at point R.

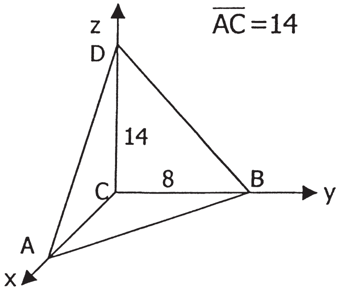
38. A ray of light, starting at the light source Q(20/6/0) travelling in the direction of P(12/2/2), is reflected in the xz - plane at point R and arrives then on the sphere $\mathcal{S}$: $x^2 + y^2 + z^2 - 8y - 10z + 20 = 0$ at point S. Calculate the coordinates of R and S.

39. Determine the equation of the insphere of the pyramid ABCD.

a)



b)



10. VECTOR PRODUCT AND SCALAR TRIPLE PRODUCT

a) The cross product

1. The vectors $\vec{a} = \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix}$ are given.

   Show that the vector $\vec{c} = \vec{a} \times \vec{b} = \begin{pmatrix} a_y b_z - a_z b_y \\ a_z b_x - a_x b_z \\ a_x b_y - a_y b_x \end{pmatrix}$ is normal to $\vec{a}$ as well as to $\vec{b}$.

2. Similarly for $\vec{a} = \begin{pmatrix} -3 \\ 5 \\ 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$.

3. Prove the following laws:

   a) $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$ b) $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

   c) $(u\vec{a}) \times (v\vec{b}) = uv(\vec{a} \times \vec{b})$; u and v are real numbers

4a) Calculate $\vec{a} \times \vec{a}$, analyse your answer.

 b) When does the following hold $\vec{a} \times \vec{b} = \vec{0}$?

5. Calculate: a) $(\vec{a} - \vec{b}) \times (\vec{b} - \vec{a})$ b) $(\vec{a} - 2\vec{b}) \times (\vec{a} + 2\vec{b})$

   c) $(3\vec{a} + \vec{b}) \times (3\vec{a} + \vec{b})$ d) $(4\vec{a} - 3\vec{b}) \times (3\vec{a} - 4\vec{b})$

6. Given the vectors $\vec{a}$ and $\vec{b}$ and $\vec{c} = \vec{a} \times \vec{b}$, the following holds:

   $c = ab \cdot \sin\varphi$, with $\varphi = \sphericalangle(\vec{a}, \vec{b})$.

   Demonstrate this using vectors $\vec{a} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

7. The base vectors $\vec{e}_x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\vec{e}_y = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are given.

   Evaluate $\vec{e}_x \times \vec{e}_y$, $\vec{e}_x \times \vec{e}_z$, $\vec{e}_y \times \vec{e}_z$, $\vec{e}_z \times \vec{e}_x$; before you calculate or guess an answer.

8. The points A(-5/3/3), B(1/5/8) and C(7/11/9) are given.
   a) Calculate $\vec{u} = \overrightarrow{AB} \times \overrightarrow{AC}$, $\vec{v} = \overrightarrow{AB} \times \overrightarrow{BC}$, $\vec{w} = \overrightarrow{AC} \times \overrightarrow{BC}$.
   b) Show that $\vec{u} \cdot \overrightarrow{AB} = \vec{u} \cdot \overrightarrow{AC} = 0$.
   c) Compute the length of the vectors $\vec{u}$, $\vec{v}$, $\vec{w}$; interpret the result geometrically.

9. Determine the Cartesian equation of the plane ABC with the aid of the cross product.

   a) A(5/5/-3), B(1/-1/0), C(2/0/-1)
   b) A(1/4/4), B(-2/-4/4), C(0/1/3)
   c) A(5/3/4), B(-8/-6/-2), C(7/6/6)
   d) A(-3/1/4), B(2/-1/-3), C(4/-7/2)

10. Determine a vector equation for the line of intersection of the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ with the aid of the vector product.

    a) $\mathcal{P}_1$: $2x - y + 3z - 5 = 0$, $\mathcal{P}_2$: $x - y + 5z + 2 = 0$
    b) $\mathcal{P}_1$: $3x + 7y + 3z - 7 = 0$, $\mathcal{P}_2$: $x + 4y + z + 1 = 0$
    c) $\mathcal{P}_1$: $3x - 4y + z = 0$, $\mathcal{P}_2$: $5x - 2y + 3z - 10 = 0$
    d) $\mathcal{P}_1$: $2y + 5z - 4 = 0$, $\mathcal{P}_2$: $3x + y - 4z - 11 = 0$

11. Show in different ways that A(7/-2/15), B(9/4/7) and C(2/-17/35) lie on the same line.

12. Calculate the area of the triangle ABC.

    a) A(7/-3/1), B(2/0/5), C(9/-3/1)
    b) A(5/2/-8), B(7/8/13), C(11/8/11)
    c) A(-2/11/23), B(1/7/-37), C(1/8/-41)
    d) A(-2/1/3), B(3/5/6), C(-1/5/2)

13. The planes $\mathcal{P}_1$: $3x - 5y + z - 4 = 0$ and $\mathcal{P}_2$: $2x - 4y + z + 7 = 0$ as well as point P(5/2/-3) are given.
    a) Find the Cartesian equation of the plane that contains P and is normal to $\mathcal{P}_1$ and $\mathcal{P}_2$.
    b) Find a vector equation of the line through P, which intersects neither plane $\mathcal{P}_1$ nor plane $\mathcal{P}_2$.

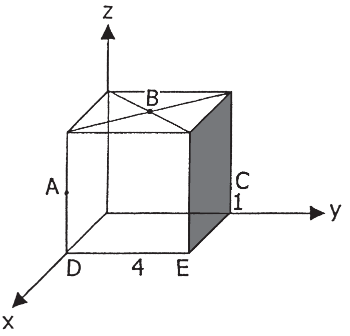
14. Show that $\tan\varphi = \dfrac{|\vec{a} \times \vec{b}|}{\vec{a} \cdot \vec{b}}$ given that $\varphi = \sphericalangle(\vec{a}, \vec{b})$.

15. Let d be the distance from a point P to the line passing through A and B.
    Show that $d = \dfrac{|\overrightarrow{AB} \times \overrightarrow{AP}|}{\overline{AB}}$.

16. With the aid of the formula in exercise 15, determine the distance from the point P to the line AB.
    a) A(2/3/0), B(1/-3/-2); P(5/8/4)   b) A(1/2/-1), B(4/4/-3); P(0/3/7)
    c) A(10/-2/7), B(-6/0/-5); P(8/5/-6)   d) A(2/1/-2), B(-2/1/0); P(4/0/7)

17. The points A(8/-2/3), B(0/5/-3), C(-4/7/-6) and D(16/1/-16) are given. Calculate the area of the triangle ABC and the volume of the pyramid ABCD.

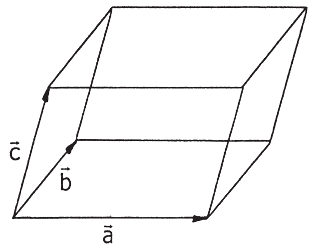
18. A is the midpoint of the edge of the cube. (☞ right).
    a) Compute the area of the triangle ABC.
    b) A point P lies on the edge DE. Find the coordinates of P given that the area A of the triangle APB is $2\sqrt{6}$.



b) The scalar triple product

19. A parallelepiped is made up of the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

    Show that the volume V is given by $V = \left|\left(\vec{a} \times \vec{b}\right) \cdot \vec{c}\right|$.



20. A parallelepiped is given by the vectors $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$ and $\vec{c} = \overrightarrow{OC}$. Find its volume (☞ O: origin).

    a) A(1/1/4), B(2/5/-4), C(1/2/1)   b) A(5/3/0), B(-7/-2/4), C(0/5/2)
    c) A(0/7/3), B(3/0/7), C(7/3/0)   d) A(2/-1/-5), B(3/8/4), C(-2/2/9)

21. Determine whether the vectors $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$ and $\vec{c} = \overrightarrow{OC}$ are coplanar (i.e. parallel to the same plane). O is the origin.

    a) A(4/0/-1), B(2/5/3), C(6/5/2)
    b) A(-7/-3/4), B(8/9/-5), C(3/5/-2)
    c) A(0/9/-8), B(12/13/8), C(6/5/8)
    d) A(3/4/5), B(5/4/3), C(4/4/4)

22. The vectors $\vec{a} = \overrightarrow{OA}$, $\vec{b} = \overrightarrow{OB}$ and $\vec{c} = \overrightarrow{OC}$ are coplanar; O is the origin. Determine the missing coordinates.

a) A(3/-3/1), B(1/4/-1), C(x/3/-2)
b) A(-5/-7/8), B(-3/y/4), C(4/2/-6)
c) A(-2/-2/3), B(2/-3/4), C(x/x/x – 5)
d) A(-5/3/z), B(7/-4/z+1), C(-10/6/z+2)

23. Are the points A, B, C and D in the same plane ?

a) A(4/1/-1), B(3/10/2), C(-2/-1/3), D(2/-1/0)
b) A(0/2/9), B(-3/5/10), C(3/-3/2), D(-2/4/11)
c) A(4/-5/2), B(1/0/-1), C(-1/4/-2), D(2/1/4)

24. A parallelepiped is defined by $\vec{a}$, $\vec{b}$ and $\vec{c}$. Its volume is V. Determine the unknown values (☞ there are always two answers).

a) $\vec{a} = \begin{pmatrix} x \\ 3 \\ 1 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 0 \\ -2 \\ 4 \end{pmatrix}$, V = 6

b) $\vec{a} = \begin{pmatrix} -2 \\ 3 \\ 8 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ y \\ -4 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 0 \\ -7 \\ 2 \end{pmatrix}$, V = 26

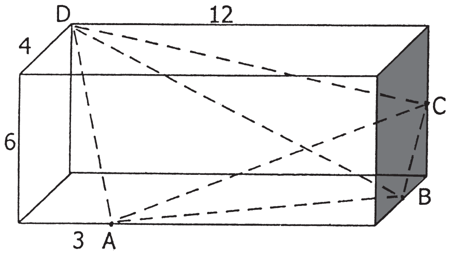
c) $\vec{a} = \begin{pmatrix} 12 \\ 7 \\ 3 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} -8 \\ 2 \\ -5 \end{pmatrix}$, $\vec{c} = \begin{pmatrix} 8 \\ 6 \\ z \end{pmatrix}$; V = 32

25. Find a formula for the volume of a tetrahedron ABCD.

26. Determine the volume of the tetrahedron ABCD.

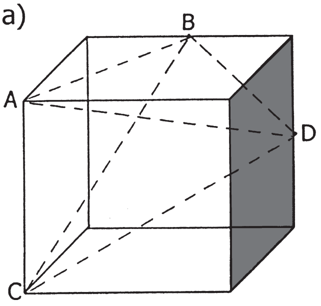
a) A(5/-2/1), B(7/4/3), C(-6/1/0), D(3/1/4)
b) A(7/9/3), B(12/-7/-5), C(-8/11/7), D(-13/10/8)
c) A(0/4/0), B(21/7/0), C(-5/0/13), D(7/2/-3)

27. Given a cuboid where the length of the sides are 4, 6 and 12 (☞ right) and where B and C are midpoints of edges. Find the volume of the tetrahedron ABCD.

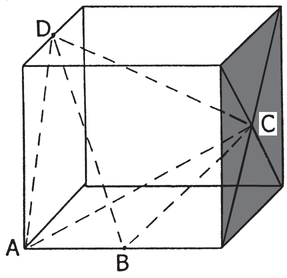


28. B and D are midpoints of the edges of a cube (☞ below).
What fraction of the total volume of the cube is the volume of the pyramid ABCD ?

a)



b)



29. The plane $\mathscr{P}$: $x - 2y + 3z + 7 = 0$ and the points A(-3/5/-1), B(-1/3/0), C(4/1/3), D(7/1/5), E(-5/-2/10) and F(-3/-1/7) are given.
   a) Show that the lines ℓ_1 = AB, ℓ_2 = CD and ℓ_3 = EF intersect at one point S. Determine the coordinates of S.
   b) The three lines intersect the plane $\mathscr{P}$ at the points P, Q, R. Find these point.
   c) Calculate the volume of the pyramid PQRS.

30. The pyramid ABCD has volume V.
Determine the missing coordinates (☞two answers in each case).
   a) A(3/8/3), B(-2/5/0), C(7/6/-1), D(x/4/1); V = 11
   b) A(4/-4/1), B(0/3/7), C(5/y/-5), D(3/-4/7); V = 9

11. EXERCISES IN VECTOR GEOMETRY FOR THE MATURA-EXAM

1. The points A(6/18/8), B(5/16/5), the plane $\mathcal{P}$: $3x - y+2z - 2 = 0$ and the sphere $\mathcal{S}$: $x^2 + y^2 + z^2 - 8x - 10y - 2z + 2 = 0$ are given. A ray of light ℓ starting at A and travelling in the direction of B is reflected in the plane $\mathcal{P}$ and then arrives on the sphere $\mathcal{S}$ at P.
   a) Determine the centre O and the radius r of the sphere $\mathcal{S}$.
   b) Find the angle of incidence of ℓ at the plane $\mathcal{P}$.
   c) What are the coordinates of point P ?

2. The sphere $\mathcal{S}$: $x^2 + y^2 + z^2 - 2z - 8 = 0$ and the plane $\mathcal{P}$ which passes through the points A(-1/9/8), B(1/10/10) and C(-5/5/8) are given.
   a) Find the point on the sphere $\mathcal{S}$ which has the shortest distance to plane $\mathcal{P}$. Determine this distance.
   b) Determine the equation of the smallest sphere that touches sphere $\mathcal{S}$ and plane $\mathcal{P}$.
   c) Determine the equations of those tangent planes at sphere $\mathcal{S}$, which are normal to the line through A and B.

3. The plane $\mathcal{P}$ is given by points A(0/-10/9), B(-4/-2/1) and P(-2/8/-2). Line ℓ contains the point B and is parallel to the y - axis. The isosceles triangle ABC has $\overline{AB}$ as the base; the vertex C lies on the line ℓ.
   a) Determine the coordinates of C as well as the angles and the area of the triangle ABC.
   b) The plane $\mathcal{P}$ and the reflected triangle ABC together form a pyramid ABCC'. Determine the coordinates of C' as well as the volume of the pyramid ABCC'.

4a) The sphere $\mathcal{S}$ with centre O(7/8/7) touches the plane $\mathcal{P}$: $x + z = 0$. Find the equation of the sphere and the coordinates of the point of contact.
 b) The point O is the vertex of a right pyramid with the rectangle ABCD as the base. Given the vertices A(7/1/0) and B(4/0/2) and that vertex C lies on the y-axis, determine the coordinates of C and D as well as the volume of the pyramid.
 c) The plane ABCD intersects the sphere $\mathcal{S}$ in a circle. Determine the centre and the radius of the circle.

5. The plane $\mathcal{P}$ which contains the points A(3/-7/12), B(-1/1/4) and C(1/11/1) and the line ℓ which passes through the point B and which is parallel to the y - axis are given.

   a) Determine the equation of the sphere S whose centre lies on the line ℓ and which contains the points A and B.
   b) Determine the equation of the sphere which is concentric to the sphere S and which touches the plane $\mathcal{P}$. Find the coordinates of the point of contact.
   c) Find the equation of the largest sphere which touches plane $\mathcal{P}$ and sphere S from the inside.

6. The points A(12/10/0), B(9/7/12) and C(-2/2/8) are given.

   a) Show that ABC is an isosceles right - angled triangle.
   b) Calculate the coordinates of the point D given that ABCD forms a square.
   d) Determine the coordinates of the vertex V of the right square - based pyramid ABCDV with the volume 1944 (☞ two answers).

7. Given the line ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 9 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}$ as well as the points A(-1/-1/-2) and B(7/3/6),

   a) find the equation of a sphere S whose centre B touches the plane $\mathcal{P}$ which contains A and ℓ.
   b) Show that the triangle ABC is isosceles for every point $C \in \ell$.
   c) Determine the point $C \in \ell$ given that the triangle ABC is right - angled.

8. The plane $\mathcal{P}_1$: 2x - 2y - z - 12 = 0 is given. The plane $\mathcal{P}_2$ is given by the points A(1/2/4), B(2/2/6) and C(3/5/2). The line ℓ passes through the points P(15/16/13) and Q(1/2/-1).

   a) Show that the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ are parallel.
   b) Determine the distance between the two planes.
   c) A ray of light ℓ starting in P and travelling in the direction of Q is reflected in the plane $\mathcal{P}_1$. Where do the incoming ray of light ℓ and the reflected ray of light ℓ' arrive on the plane $\mathcal{P}_2$?
   d) Determine the equation of the sphere S that touches the planes $\mathcal{P}_1$ and $\mathcal{P}_2$ and whose centre lies on the line ℓ.

9. The point P(5/1/0) lies on the sphere S_1: $x^2+y^2+z^2+6x - 14y+a = 0$; the line ℓ is defined by the points A(7/2/-2) and B(11/5/-4).
   a) Determine the centre O_1 and the radius r_1 of the sphere S_1.
   b) Show that the line ℓ is a tangent of the sphere S_1.
   c) Find the distance from the centre O_1 of the sphere S_1 to the plane $\mathcal{P}$, defined by the line ℓ and the point P.
   d) Determine the equation of the sphere S_2 with radius $r_2 = 5$ that touches the sphere S_1 at the point P, given that the centre of S_2 lies outside of S_1.
   e) Determine the equation for the common tangent t of S_1 and S_2 that passes through the point P and lies in the plane $\mathcal{P}$.

10a) The vectors $\vec{p} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $\vec{q} = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ are perpendicular.
   The length of vector $\vec{p}$ is 15. Furthermore: $x + 5z = 0$, $x > 0$.
   Determine the components of vector $\vec{p}$.
   b) With the vectors $\overrightarrow{AB} = \vec{p}$ and $\overrightarrow{AD} = c \cdot \vec{q}$, $c > 0$, a square ABCD is constructed starting at A(-2/1/7).
   Determine c and the coordinates of the vertices B, C and D.
   d) The origin O is the vertex of the pyramid with base ABCD.
   Calculate the volume of the pyramid.

11. Given the circle c: $x^2 + y^2 - 10x - 14y + 49 = 0$ in the xy - plane as well as the point P(0/-3/0),
   a) find the equations of the tangents to the circle c through the point P.
   b) Find the equation of the sphere S, which touches the plane $\mathcal{P}$: $z = 25$ and intersects the xy - plane in the circle c.
   c) Find the Cartesian equation of the tangent plane to the sphere S that contains the y - axis.

12. The line ℓ through the points A(4/3/4) and B(5/4/4) and the sphere S_1 with centre $O_1(-2/13/0)$ and radius $r_1 = 6$ are given.
   a) What are the coordinates of the point P on the line ℓ with the shortest distance from the sphere S_1?
   b) Find the equation of the smallest sphere S_2 that touches the line ℓ and the sphere S_1.
   c) Find the equation of the common tangent plane to the two spheres through the point of contact.

Answers

1. Trigonometry

1a) 0.242, 0.605, ... b) 0.998, 0.150, ... c) 0.424, -2.450, ...
d) 53.13°, 18.06°... e) 60°, 39.65°, ... f) 11.31°, 63.43°, ...

2a) 0.707, 0.866, ... b) 0.866, 0, ... c) 0.577, undefined, ... 1

3a) 45° b) 150° c) 57.30° d) 30.94° e) 332.89° f) 1145.92°

4a) 1.75 b) 7.75 c) 0.17 d) 12.34 e) 15.71

5. $-1 \le \sin\alpha \le 1$, $-1 \le \cos\alpha \le 1$, $-\infty < \tan\alpha < +\infty$

6.

	a) 0°	b) 30°	c) 45°	d) 60°	e) 90°	f) 180°
$\sin\alpha$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\alpha$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\alpha$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	---	0

7a) 6.15° b) 20.44° c) 38.94° d) 48.97° e) 50.12° 2

8a) 3.08° b) 28.08° c) 77.76° d) 82.27° e) 89.40°

	a	b	c	α (in °)	β (in °)
9a)	--	--	29	43.60	46.40
b)	--	105	--	39.97	50.03
c)	--	14.30	18.67	--	50
d)	1.21	--	5.93	11.8	--
e)	22.18	24.03	--	42.7	--
f)	7.10	2.94	--	67.5	22.5
10a)	16.47	--	24.77	41.67	48.33
b)	6.64	15.13	16.52	--	66.3
c)	50.48	75.72	91	33.69	56.31
d)	--	33.24	35.51	20.61	69.39
e)	14.62	40.85	43.39	19.7	--
f)	16.53	10.9	19.80	56.60	33.40
11a)	--	26.83	38.64	46.01	43.99
b)	6.71	4.47	8.06	56.31	33.69
c)	30	12.5	32.5	67.38	22.62
d)	15	20	25	36.87	53.13

12a) $w_\alpha \approx 70.97$, $w_\beta \approx 29.49$
b) 225.15 resp. 25.16
c) 23.58
d) 219.69
e) 15.51
f) 18.92

13a) $\beta \approx 73.74°$, $\gamma \approx 32.52°$
b) $a \approx 16.52$, $\gamma = 54°$
c) $c \approx 11.62$, $\beta \approx 81.71°$, $\gamma \approx 16.58°$
d) $a \approx 9.51$, $\beta \approx 67.16°$, $\gamma \approx 45.68°$
e) $a \approx 44.01$, $c \approx 37.89$, $\beta = 64.5°$
f) $a \approx 65.04$, $c \approx 62.26$, $\beta = 61.4°$ 3

14. $w_\beta \approx 77.14$

15. 63.78°

16. 110.01° resp. 69.99°

17. 9.75 resp. 5.74

18a) $c \approx 16.34$, $h \approx 8.14$
b) $a \approx 154.17$, $\alpha \approx 16.18°$
c) $h \approx 26.43$, $\alpha \approx 78.25°$
d) $a \approx 38.37$, $b \approx 14.76$

19. 85.71°

20a) 54.74°
b) 45° resp. 90° 4
c) 35.26°

21. 52.13° 22. 2.00 23a) 97.18° b) 162.36° 24. A≈1.72, d≈1.62

4

25a) 1220.21
b) 1275.34

26a) 30.67
b) 28.67
c) 95.46

27. 16.76
28. 674.10
29. 43.72

30. 4.00
31. ε≈35.26°, x≈0.41

5

32. ε=30°, x= $\sqrt{3}$ 33. 36.87° 34. 11.44

6

35a) (3/90°)
b) (7/0°)
c) (4/180°)
d) (7.07/45°)
e) (8/120°)
f) (13/292.62°)
g) (7.62/113.20°)
h) (10/233.13°)
i) (25/106.26°)

36a) (5/0)
b) (0/-3)
c) (-4/-4)
d) (-2/3.46)
e) (-1.5/-2.60)
f) (-1.12/4.87)
g) (0.21/-1.99)
h) (-1.46/-6.85)
i) (14.49/-3.88)

37a) 35 m
b) 22°

38a) β≈31.60°; δ=α
b) 41 mm

39. 512 m
40. 41 m

41a) approx. 10°
b) approx. 1100 km

7

42. Conclusion from the definition of the trigonometric functions

43a) 11.54°, 168.46°
b) 227.73°, 312.27°
c) 32.86°, 327.14°
d) 92.87°, 267.13°
e) 87.27°, 267.27°
f) 152.98°, 332.98°
g) 154.16°, 205.84°
h) 116.57°, 296.57°
i) 15.66°, 164.34°
k) 50.19°, 230.19°
l) 89.83°, 270.17°
m) 184.01°, 355.99°

44a) b≈20.03, c≈26.05, γ=95°
b) a≈6.64, c≈5.17, β=87°
c) a≈11.69, b≈2.03, α=76,5°
d) b≈273.94, c≈308.09, β=46°

45a) c≈34.36, α≈34.02°, γ≈73.98°
b) a≈11.91, α≈87.81°, γ≈40.89°
c) 1. b≈76.49, α≈20.20°, β≈141.90°
2. b≈4.97, α≈159.80°, β≈2.30°
d) unsolvable (no triangle)

46a) c≈8.26, α≈69.24°, β≈35.76°
b) b≈7.74, α≈81.70°, γ≈47.00°
c) a≈79.12, β≈48.93°, γ≈21.77°
d) b≈70.99, α≈3.51°, γ≈0.19°

47a) α≈44.41°, β≈57.12°, γ≈78.46°
b) α≈37,18°, β≈93,18°, γ≈49.64°
c) α≈33.82°, β≈93.35°, γ≈52.84°
d) unsolvable (no triangle)

48a) r≈9.05, A≈94.87
b) r≈20.50, A≈384.94

49. α≈81.54°, β≈17.17°; γ≈81.30°

50. α≈33.20°, β≈87.80°, b≈41.97, c≈36.00

8

51a) 5.57
b) 55.43
c) 14.94
d) 94.49

52. 71.89

53. $h_a \approx 8.65$; $s_a \approx 11.12$; $w_\alpha \approx 10.50$

54a) $\sqrt[3]{2} \approx 1.2599$
b) $e_3 \approx 1.2593$; $e_4 \approx 1.2605$

55a) $h_a \approx 16.73$, $h_b \approx 11.81$, $h_c \approx 10.57$
b) $h_a \approx 44.80$, $h_b \approx 104.84$, $h_c \approx 38.41$

56a) 57.69°
b) 4.73

9

57. 8.15 58. 19.97°, 160.03° 59. e≈61.48; f≈33.17

60a) b≈33.19, f≈48.48, β≈50.28°
b) a≈57.97, d≈22.27, f≈45.78
c) a≈20.26, f≈14.79, α=37°
d) a≈28.64, c≈20.43, β=70°

61. 44.69

62a) 38.50
b) 20.43°

63. 7.18

64a) 1.52
b) 2.23

9

65. 3.70 (radii 3, 6 and 7)   66. x≈3.59, y≈4.96   67. 12.75

10

68. α≈57,43°, β≈86,82°, γ≈35,75° (Hint: centre of gravity theorem)

69. β=30°, γ=105° (Hint: similar triangles)   70. 56 m   71. 169 m

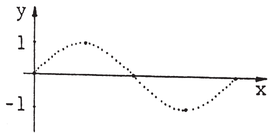
72. 228 m   73. 583 m   74. 408 m   75. 154 km

11

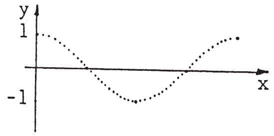
76. 36 sm, i.e. 67 km
77a) 12 km
b) N65W
78. 17.8 sm, i.e. 33 km
79. 478 km/h; N34E

12

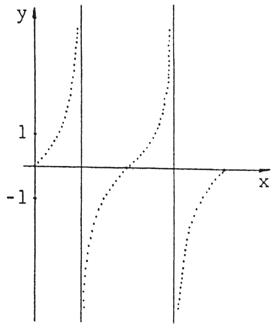
80a)



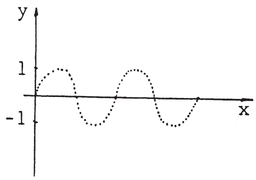
80b)



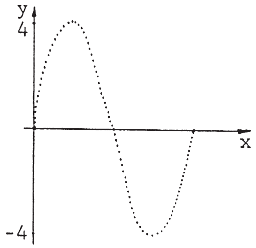
80c)



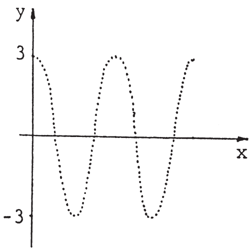
81a)



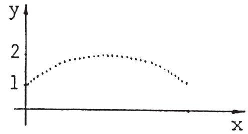
81b)



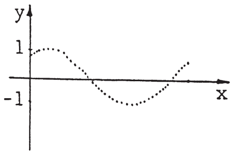
81c)



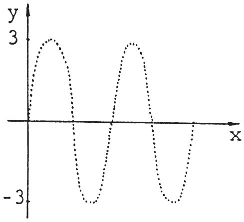
81d)



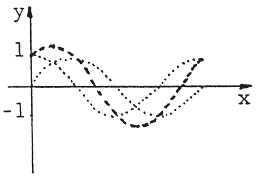
82a)



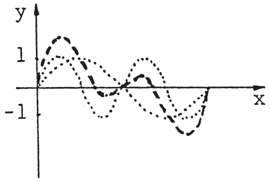
82b)



83a)



83b)



12

84a) $\pi/2$, $3\pi/2$
b) $0, \pi/3, 2\pi/3, \ldots, 2\pi$
c) $3\pi/2$
d) $0, \pi/2, \pi, 3\pi/2, 2\pi$
e) $0, \pi, 2\pi$
f) $\pi/2, 3\pi/4, 3\pi/2, 7\pi/4$

85a) $2\pi/5$
b) $2\pi/9$
c) $\pi/8$
d) $8\pi/3$
e) 3π
f) 4π

86a) $[-1, 1]$
b) $[-2, 2]$
c) $(-\infty, +\infty)$
d) $[-1, 1]$
e) $[-1, 1]$
f) $(-\infty, +\infty)$

13

87a) $\pi < x < 2\pi$
b) $0 \le x < \pi/3 \vee 5\pi/3 < x \le 2\pi$
c) $0 \le x < \pi/4 \vee \pi/2 < x < 5\pi/4 \vee 3\pi/2 < x \le 2\pi$
d) $\pi/4 < x < 7\pi/4$
e) $\pi/3 < x < \pi/2 \vee 4\pi/3 < x < 3\pi/2$
f) $\pi/4 < x < 3\pi/4$

88. -

89a) $\cos\alpha$
b) $1/\tan\alpha$
c) $\sin\alpha$
d) $\sin^2\alpha - \cos^2\alpha$

90a) $\cos\alpha = 0.96$, $\tan\alpha = 7/24$
b) $\sin\alpha = 5/13$, $\tan\alpha = 5/12$
c) $\sin\alpha = 0.6$, $\cos\alpha = 0.8$
d) $\cos\alpha = 8/17$, $\tan\alpha = 15/8$
e) $\sin\alpha = 20/101$, $\tan\alpha = 20/99$
f) $\sin\alpha = 35/37$, $\cos\alpha = 12/37$

91a) $\sin(\alpha+\beta) = 156/205$, $\sin((\alpha-\beta) = 84/205$, $\cos(\alpha+\beta) = 133/205$,
$\cos(\alpha-\beta) = 187/205$, $\tan(\alpha+\beta) = 156/133$, $\tan(\alpha-\beta) = 84/187$
b) $\sin(\alpha+\beta) = 171/221$, $\sin(\alpha-\beta) = -21/221$, $\cos(\alpha+\beta) = -140/221$,
$\cos(\alpha-\beta) = 220/221$, $\tan(\alpha+\beta) = -171/140$, $\tan(\alpha-\beta) = -21/220$

92. Application of the addition theorems

93a) $2\sin\alpha \cdot \cos\beta$
b) $2\cos\alpha \cdot \sin\beta$
c) $1+\tan\alpha \cdot \tan\beta$
d) $1 - \tan^2\alpha$
e) $2\sin^2\alpha$
f) 2
g) $\sin\alpha$
h) $\cos\alpha$

14

94a) ± 0.96
b) -0.02
c) -0.75
d) 0.296
e) 0.8432
f) $\sqrt{3}/2$

95a) $-3 \le a \le 3$
b) $-1 \le a \le 7$
c) $a \ge 5 \vee a \le -5$
d) $a \ge -0.5$
e) $-1.25 \le a \le 1.25$
f) $a \ge 0.2 \vee a \le -1$

96a) 47.05°, 112.95°
b) 180°, 198°
c) 71.74°, 251.74°
d) 30.21°, 113.79°
e) 56.58°, 146.58°, 236.58°, 326.58°
f) 74.21°, 351.79°

97a) 45°, 135°, 225°, 315°
b) 0°, 60°, 120°, 180°, 240°, 300°
c) 45°, 225°
d) 153°, 333°
e) 7.5°, 187.5°
f) 122°, 302°

98a) 45°, 225°
b) 0°, 90°, 180°, 270°
c) 0°, 180° **14**

99a) 34.99°, 214.99°
b) 71.57°, 251.57°
c) 54.46°, 234.46°

100a) 0°, 180°
b) 0°, 180°
c) 0°, 45°, 180°, 225°
d) 0°, 30°, 150°, 180°

Exercises 101 to 106: quadratic equations

101a) 120°, 240°
b) 30°, 150°

102a) 68.53°, 291.47°
b) 90°, 270°
c) 0°, 90°, 180°,270°
d) 45°, 90°, 135°, 225°, 270°, 315°

103a) 30°, 150°, 210°, 330°
b) 0°, 54.74°, 125.26°, 180°, 234.74°, 305.26°

104a) 60°, 180°, 300°
b) 45°, 123.69°, 225°, 303.69°
c) { }
d) 70.53°, 289.47°

105a) 39.23°, 140.77°, 219.23°, 320.77°
b) 30°, 150°, 210°, 330°

106a) 90°
b) 270°

107. $A \approx 47.90$, $p \approx 31.65$

108. $A \approx 89.83$

109a) $p \approx 64.87$
b) 57.93° resp. 230.07°

110a) 55.77°
b) $A \approx 7.91$ **15**

111a) 20.02
b) 329.89 (Hint: Decomposition in triangle and sector)

112a) $\overline{AD} = 4$, $\overline{AS} \approx 7.16$, $\overline{DS} \approx 6.45$; angle 63.43°, 33.690°, 82.88° **16**
b) A = 12.8, i.e. exactly 40% of the area of the rectangle

113. $\overline{EM} \approx$ 366'094 km (Hint: Determine triangle BEC first; $\overline{BC} \approx$ 8724 km)

2. Oblique parallel projection

For each exercise, the solution to part a) is illustrated.

1a) $S_1(2/4/0)$, $S_2(0/5/-1)$, $S_3(10/0/4)$; ☞ figure 1, page 64
b) $S_1(8/-3/0)$, S_2 (0/9/4), $S_3(6/0/1)$
c) $S_1(-2/3/0)$, S_2 (0/2/1), $S_3(4/0/3)$

2a) $S_1(-2/10/0)$, $S_2(0/8/1)$, $S_3(8/0/5)$; visible $\overline{S_2S_3}$; ☞ figure 2 **17**
b) $S_1(6/3/0)$, $S_2(0/5/2)$, $S_3(15/0/-3)$; visible $\overline{S_1S_2}$

3a) $\overline{AB}$ visible; parallel: S_1(3/2/0), $S_2(0/3/3)$,
visible $\overline{S_1S_2}$; ☞ figure 3
b) AB: $S_1(4/6/0)$, S_3 (2/0/4); visible $\overline{S_1S_3}$
parallel: $T_1(7/3/0)$, $T_3(6/0/2)$; visible $\overline{T_1T_3}$

17

4a) $S_1(4/4/0)$ on AC; $T_1(6/6/0)$ on AB; visible $S_1T_1BCS_1$; ☞ figure 4

b) $S_1(8/2/0)$, $S_3(7/0/2)$ on AB; $T_1(6/6/0)$ on BC; $U_3(5/0/4)$ on AC; visible $S_3S_1T_1CU_3S_3$

c) $S_1(4/3/0)$ and $S_3(3/0/2)$ on AB; $T_3(1/0/2)$ and $T_{12}(0/3/0)$ on BC; $U_3(5/0/4)$ on AC; visible $S_1S_3T_3T_{12}S_1$

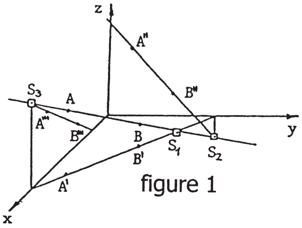


figure 1

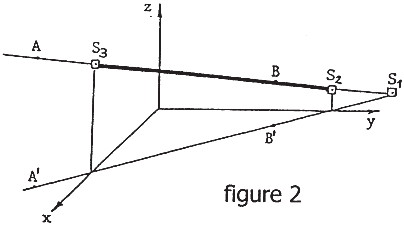


figure 2

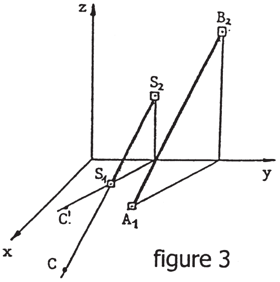


figure 3

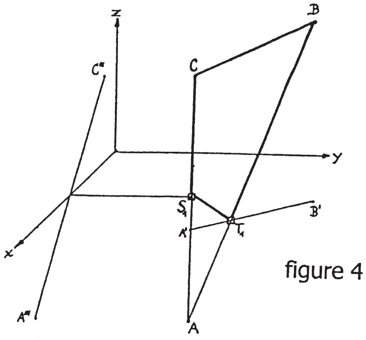


figure 4

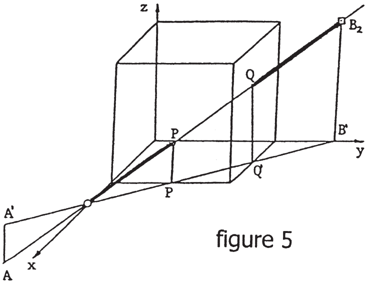


figure 5

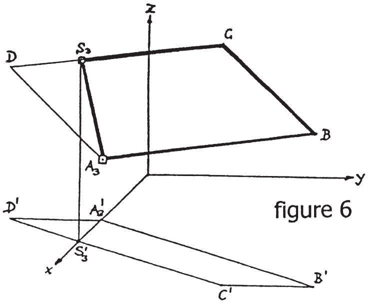


figure 6

18

5a) AB: $S_{13}(9/0/0)$, $B=S_2$; intersection point P(6/3/2), Q(3/6/4); ☞ fig. 5 (above)

b) AB: $S_2(0/5/8)$, $S_3(10/0/3)$; intersection point P(6/2/5), Q(4/3/6); use the side projection

c) AB: S_1 (4/9/0), $B=S_3$; intersection point P(3/6/3), Q(2/3/6); use the vertical or the side projection.

6a) D(4/-3/5); $S_3(6/0/6)$ on $\overline{CD}$; visible quadrilateral $ABCS_3$ ☞ figure 6 (above)

b) D(5/1/3); $S_1(6/10/0)$ on $\overline{AB}$, $T_2(0/8/5)$ on $\overline{BC}$, $U_2(0/6/5.5)$ on $\overline{CD}$, $V_1(8/4/0)$ on $\overline{AD}$; visible hexagon $BT_2U_2DV_1S_1$

7. The axes intercepts are indicated.

a) x=12, y=9, z=6; ☞ figure 7
b) x=6, y=4, z= - 6
c) x=14, y=7, (z=42)
d) z=4; 1. principal plane
e) x=12, z=6; 3. projecting plane
f) x=4, (y= - 4), z=2
g) (x=24), y=8, z=6
h) x=4, y=4, (z= - 2); help point necessary
i) x=8, (y=40), z=5; help point necessary

8a) ABC: intercepts x=12, y=6, z=3; DEF: intercepts x=6, y=3, z=6
intersection line S_1S_2 with $S_1(4/0/2)$, $S_2(0/2/2)$; ☞ figure 8

b) ABC: x=12, y=9, z=6; DEF: x=2, y=4, (z= - 4)
intersection line S_2S_3 with $S_2(0/6/2)$, $S_3(4/0/4)$

c) ABC: x=12, y=6 (1. projecting); DEF: y=4, z=6 (2. projecting)
intersection line S_1S_3 with $S_1(4/4/0)$, $S_3(12/0/6)$

d) ABC: z=3 (1. principal plane); DEF: (x= -6), y=6, z=2
intersection line S_2S_3 with $S_2(0/-3/3)$, $S_3(3/0/3)$

9a) ABC: x=12, y=8, z=6; PQ: visible QS with S(3/2/3); ☞ figure 9 (below)

19

b) ABC: y=8, z=4 (2. projecting); PQ: visible S_3S with $S_3(4/0/6)$, S(8/4/2)

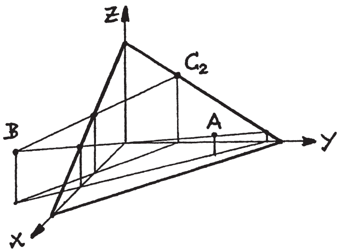
c) ABC: x=12, (y=24), z=4; PQ: visible S_3S with $S_3(6/0/9)$, S(2/2/3)

d) ABC: x=8, y=3, (z= - 12); PQ: visible from $S_2(0/5{,}5/1)$ (to S_3)

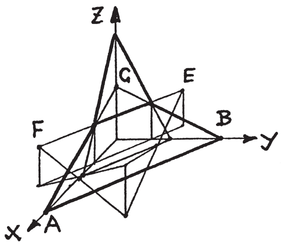
e) ABC: z=3 (1. principal plane); PQ: visible S_3S with $S_3(6/0/5)$, S(2/8/3)

10. ☞ figures 10a), b), c), d)

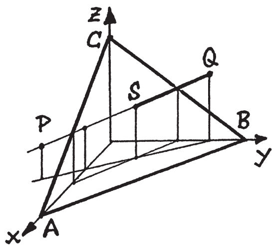
a) ABC: x=10, y=8, z=9
b) ABC: x=8, (y= - 10), z=4
c) ABC: (x= - 36), y=9, z=8
d) ABC: (x=30), y=10, z=8



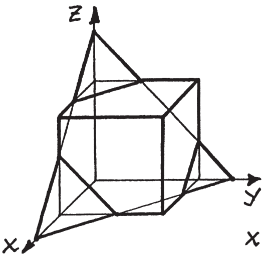
Figur 7



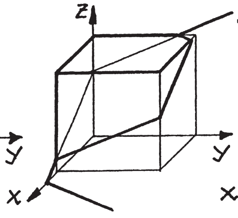
Figur 8



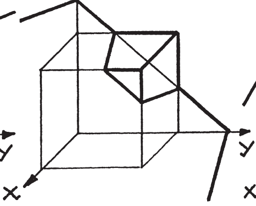
Figur 9



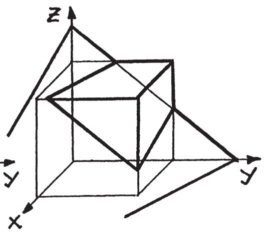
10a)



10b)

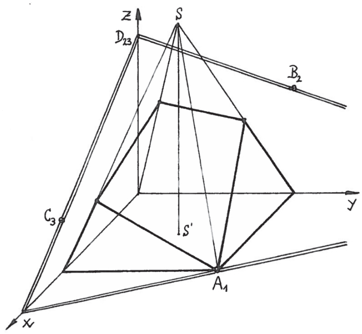


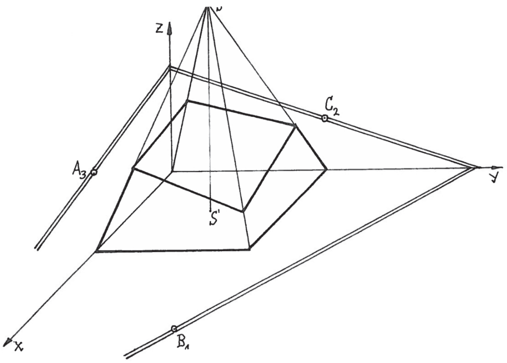
10c)



10d)

19 11. ☞ figures 11a), b)





3. Basic vector operations

20

1. ---
2. --

3a) $c \approx 2.6\,\text{cm}$
b) $d \approx 7.6\,\text{cm}$
c) $e \approx 4.0\,\text{cm}$

4a) $u \approx 10.0\,\text{cm}$
b) $v \approx 3.2\,\text{cm}$
c) $w \approx 4.3\,\text{cm}$

5a) $1.6(\vec{b} - \vec{c})$
b) $^1/_3\,(13\vec{b} - \vec{c})$
c) $^{16}/_3\,(\vec{b} - 2\vec{c})$

6. $\overrightarrow{AC} = \vec{a} + \vec{c}$; $\overrightarrow{AD} = ½\ \vec{c}$; $\overrightarrow{CD} = -(\vec{a} + ½\ \vec{c})$

7a) $\overrightarrow{AC} = \vec{a} + \vec{b}$; $\overrightarrow{AD} = \vec{b}$; $\overrightarrow{BE} = ½\,(\vec{b} - \vec{a})$; $\overrightarrow{EC} = ½\,(\vec{a} + \vec{b})$
b) $\overrightarrow{AB} = ½\,(\vec{e} - \vec{f})$; $\overrightarrow{AD} = ½\,(\vec{e} + \vec{f}) = \overrightarrow{BC}$; $\overrightarrow{BE} = ½\vec{f}$; $\overrightarrow{EC} = ½\vec{e}$

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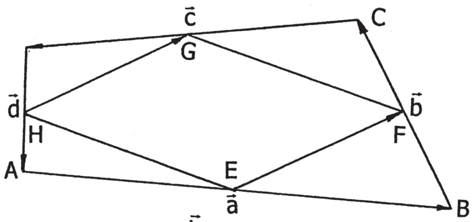
8. $\overrightarrow{AE} = ½\ \vec{a}$; $\overrightarrow{AC} = \vec{a} + \vec{b}$; $\overrightarrow{BD} = \vec{b} - \vec{a}$; $\overrightarrow{CD} = -\vec{a}$; $\overrightarrow{DE} = ½\ \vec{a} - \vec{b}$;
$\overrightarrow{BF} = {}^3/_5\,\vec{b}$; $\overrightarrow{AF} = \vec{a} + {}^3/_5\,\vec{b}$; $\overrightarrow{EF} = ½\ \vec{a} + {}^3/_5\,\vec{b}$

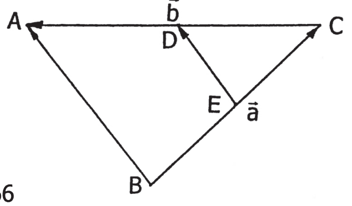
9. $\overrightarrow{AC} = \vec{a} + \vec{b}$; $\overrightarrow{BG} = \vec{b} + \vec{c}$; $\overrightarrow{AF} = \vec{a} + \vec{c}$; $\overrightarrow{EC} = \vec{a} + \vec{b} - \vec{c}$;
$\overrightarrow{AG} = \vec{a} + \vec{b} + \vec{c}$; $\overrightarrow{HF} = \vec{a} - \vec{b}$

10. $\overrightarrow{CE} = \vec{a} + \vec{b} + \vec{c}$; $\overrightarrow{FD} = \vec{b} - \vec{a} - \vec{c}$; $\overrightarrow{CI} = ½\ \vec{a} + ½\ \vec{b} + \vec{c}$;
$\overrightarrow{BK} = ½\ \vec{a} + \vec{b} - \vec{c}$; $\overrightarrow{IK} = ½\ \vec{b} - \vec{c}$

11a) $\vec{d} = -(\vec{a} + \vec{b} + \vec{c})$
b) $\overrightarrow{EF} = ½\,(\vec{a} + \vec{b})$;
$\overrightarrow{HG} = -½\,(\vec{d} + \vec{c}) = ½\,(\vec{a} + \vec{b})$
Thus: $\overrightarrow{EF} = \overrightarrow{HG}$



12. $\overrightarrow{BA} = \vec{a} + \vec{b}$
$\overrightarrow{ED} = \overrightarrow{EC} + \overrightarrow{CD} = ½\ \vec{a} + ½\ \vec{b}$
Thus: $\overrightarrow{ED} = ½\ \overrightarrow{BA}$



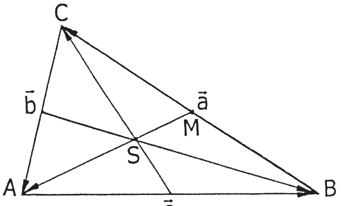
13. $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\overrightarrow{SA} = \frac{2}{3}\,\overrightarrow{MA} = \frac{2}{3}\,(\vec{b} + \frac{1}{2}\,\vec{a})$

$\overrightarrow{SB} = \frac{2}{3}(\vec{c} + \frac{1}{2}\,\vec{b})$

$\overrightarrow{SC} = \frac{2}{3}(\vec{a} + \frac{1}{2}\,\vec{c})$

$\overrightarrow{SA} + \overrightarrow{SB} + \overrightarrow{SC} = \vec{a} + \vec{b} + \vec{c} = \vec{0}$

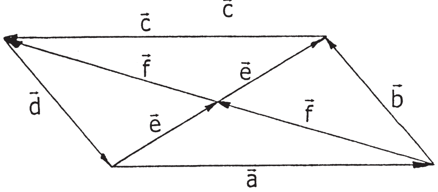


21

14. $\vec{a} + \vec{f} - \vec{e} = \vec{0}$

$\vec{c} - \vec{f} + \vec{e} = \vec{0}$

$\vec{a} + \vec{c} = \vec{0} \;\Rightarrow\; \vec{a} = -\vec{c}$



15. $\vec{m} = -\frac{1}{2}\,\vec{d} + \vec{a} + \frac{1}{2}\,\vec{b}$ (below)

$\vec{m} = \frac{1}{2}\,\vec{d} + \vec{c} - \frac{1}{2}\,\vec{b}$ (above)

Thus: $2\vec{m} = \vec{a} + \vec{c}$

4. Vectors in the coordinate system

1a) P'(5/2/-1)
b) P'(5/-2/1)
c) P'(5/-2/-1)
d) P'(-5/-2/1)
e) P'(-5/-2/-1)
f) P'(3/6/7)

2a) on x-axis
b) in xy-plane
c) in xz-plane
d) line z=4 ∥ y-axis, in yz-plane
e) plane x=3 ∥ yz-plane
f) plane y=2 ∥ xz-plane
g) line ∥ x-axis, 1./2. octant
h) line y=z in yz-plane

3a) - b) $\begin{pmatrix} 11 \\ 5 \end{pmatrix}$

22

4a) - b) $\begin{pmatrix} 6 \\ 5 \end{pmatrix}$

5a) $\begin{pmatrix} 9 \\ 20 \\ -13 \end{pmatrix}$ b) $\begin{pmatrix} 178 \\ 60 \\ 9 \end{pmatrix}$ c) $\begin{pmatrix} 69 \\ -1 \\ 8 \end{pmatrix}$

6a) no
b) yes
c) yes
d) no

7a) x=12, z = -32
b) y = 0, z = $\frac{2}{3}$
c) x = -1.25, y = 11.2
d) $\vec{a}$, $\vec{b}$ *not* collinear

23

8a) x= - 2.5
b) y=12

9. x= - 5.5
y=11.5

10a) yes
b) no

11. no

12a) D(-2/1) c) D(-4/4/1)
b) D(-5/18) d) D(14/8/-8)

13. D_1 (-2/-1/-12), D_2 (6/-5/-2), D_3 (0/9/12)

14. yes

15a) C(-6/7/11), D(0/-1/7)
b) C(4/-13/7), D(7/-6/0)

16a) $2\vec{a} + \vec{b}$
b) $\frac{1}{2}\vec{a} + 2\vec{b}$
c) $-2\vec{a} - \vec{b}$
d) $-1.5 + 3\vec{b}$

17a) $4\vec{a} - 7\vec{b} + \vec{c}$
b) $-\vec{a} - 4\vec{b} - 2\vec{c}$
c) $3\vec{a} - \vec{b}$
d) $3\vec{a} + \vec{b} - 4\vec{c}$
e) $-3\vec{a} + \vec{c}$
f) $7\vec{a} + 4\vec{b} - 3\vec{c}$

18. a=9; $\begin{pmatrix} -7.5 \\ 15 \\ 15 \end{pmatrix}$

24

19. a=25; $\begin{pmatrix} -19.2 \\ 32 \\ -14.4 \end{pmatrix}$

20a) 48
b) 84
c) 40
d) 32

25

25

21. $B_1(6/5/-3)$, $B_2\ (6/-3/-3)$
22. $P_1(0/0/5)$, $P_2(0/0/9)$
23. $P_1(8/0/0)$, $P_2(-4/0/0)$
24. $x_1 = 6, y_1 = -2\ ;\ x_2 = 2, y_2 = -6$
25. $y_1 = 6, z_1 = -3\ ;\ y_2 = -3, z_2 = 6$
26. P(3/0)

27a) P(0/-3/0)
b) P(0/17/0)
c) P(0/0/2)

28a) $P_1(4/0/0)$, $P_2(12/0/0)$
b) $P_1(0/-3/0)$, $P_2(0/3\frac{1}{2}/0)$
c) $P_1(0/0/7)$, $P_2(0/0/6\frac{1}{2})$

29a) S(1/3)
b) S(3/-2/2)

26

30a) C(-2/5/-3); $4\frac{1}{2}$
b) C(3/4/-2); $4\frac{1}{2}$

31a) P(0.5/0/0)
b) P(0/4/5)

32a) yes
b) Q(10/18/0)
c) K(0/6/5)
d) P(0/0/1)
e) P(4/6/0)

33. $A_1(0/10/0)$, $C_1(7/11/20)$; $A_2(7/11/0)$, $C_2(0/10/20)$

5. The scalar product

27

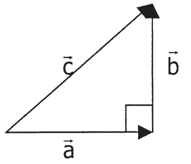
1a) 14
b) - 3
c) 12
d) 0

2a) 60°
b) 94.10°
c) 90°
d) 180°
e) 120°
f) 67.98°

3a) $\vec{a}\cdot\vec{b} = ab \cdot \cos\varphi = ba \cdot \cos\varphi = \vec{b}\cdot\vec{a}$
b) $k(\vec{a}\cdot\vec{b}) = kab \cdot \cos\varphi = (ka)b \cdot \cos\varphi =$
$= (k\vec{a})\cdot\vec{b} = (ak)b \cdot \cos\varphi = \vec{a}\cdot(k\vec{b})$
c) $\vec{a}\cdot(\vec{b}+\vec{c}) = a(b_a + c_a) = ab_a + ac_a =$
$= \vec{a}\cdot\vec{b}+\vec{a}\cdot\vec{c}$

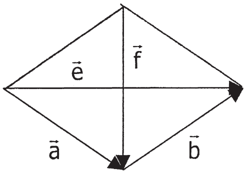
4. Follows from the definition of the scalar product resp. from the cosine

5a) 0° b) 180° c) 60° d) 120°



6a) To square $\vec{c} = \vec{a}+\vec{b}$;
$2\vec{a}\cdot\vec{b} = 0 \Rightarrow$ Pythagoras

b) Diagonals $\vec{e} = \vec{a}+\vec{b}$ and $\vec{f} = \vec{a}-\vec{b}$
$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b}) = a^2 - b^2 = 0$,
since $a = b$.
Thus $\vec{e}\cdot\vec{f} = 0$, i.e. $\sphericalangle(\vec{e},\vec{f}) = 90°$



7a) 60°
b) 60°
c) 60°
d) 75.52°
e) 123.75°
f) 120°
g) 131.81°
h) 120°
i) 104.48°

28

8a) -2
b) -56
c) 5
d) -2

9a) -7
b) -6
c) 77
d) -114
e) -77

10a) 59.49°
b) 61.93°

11a) 13.46°
b) 34.00°

12a) 68.26°
b) 86.08°

29

13a) 62.75°, 68.45°, 48.80° b) 39.63°, 90°, 50.37°

29

14a) 36.87°, 126.87°
b) 29.21°, 115.88°, 77.40°
c) 144.46°, 90°, 54.46°

15. $z = \pm 1$

16. $y = \pm 2$

17a) 11
b) $u_1 = 4$, $u_2 = -2$

18a) $P_1(0/2/0)$, $P_2(0/5/0)$
b) $P_1(4/0/0)$, $P_2(-2/0/0)$
c) $P_1(0/0/4)$, $P_2(0/0/10)$
d) $P_1(4/0/0)$, $P_2(-1/0/0)$

19. $P(0/6/12)$; $\overline{EP} = 14$

20. $A_1 = 49$ with $x_1 = -3$ and $z_1 = 3$;
$A_2 = 42.25$ with $x_2 = z_2 = 1.5$

6. The straight line

30

1a) $3x - y - 5 = 0$
b) $2x - 7y + 11 = 0$
c) $6x + 5y - 20 = 0$
d) $x + 3 = 0$
e) $y + 5 = 0$

2a) yes
b) yes

3. P yes, Q no

4a) $5x - 2y - 30 = 0$
b) $3x + y - 7 = 0$

5a) $12x - 8y + 5 = 0$
b) $8x - 12y + 21 = 0$
c) $x - 3y - 5 = 0$

31

6a) $S(-1/-11)$; 5.91°
b) $S(-3/0)$; 71.57°
c) $S(6/3)$; 90°
d) $S(-4/-8)$; 43.60°
e) $S(6/7)$; 75.96°
f) $\ell_1 \parallel \ell_2$

7. $m_1 = -2$, $m_2 = \frac{1}{2}$

8a) $P(3.75/4.5)$
b) 82.87°
c) 6.75

9a) $S(4/6)$
b) 82.87°
c) 32

10a,b) $m_1 \cdot m_2 = -1$
c) $\vec{a}_1 \cdot \vec{a}_2 = 0$

11a) $H(6/8)$
b) $H(1/1)$

12a) $P'(0/5)$
b) $P'(8/-3)$
c) $P'(10/5)$
d) $P'(-6/-8)$

13a) $3x - y - 13 = 0$
b) $2x + 29y - 12 = 0$
c) $45x + 265y - 787 = 0$

32

14a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}$

b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 5 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix}$

c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \\ 3 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

15. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1.5 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

16. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$

17. A yes, B no, C yes

18a) ℓ on z - axis
b) $\ell \parallel$ x - axis, in xz - plane
c) ℓ in xy - plane
d) $\ell \parallel$ yz - plane

19a) $S_1(1/5/0)$, $S_2(0/7/-3)$, $S_3(3.5/0/7.5)$
b) $S_1(9/1/0)$, $S_2(0/-3.5/13.5)$, $S_3(7/0/3)$

20. $S_a(-0.8/3.6/0)$, $S_b(2.5/15/0)$, $S_c(-3/-4/0)$

21. $T_1(-1/3/0)$, $T_2(2/1/2)$

22. yes

33

23a) skew
b) parallel
c) coincident
d) intersecting at S(6/4/7)
e) parallel
f) intersecting at S(0/0/-3)
g) skew

24a) S(0/0/7), 90° b) S(0/8/15), 65.44° c) S(33/44/55), 63.72°

34

25a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 5 \\ -3 \end{pmatrix} + t \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix}$, S(1/10/-7)

b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 8 \end{pmatrix} + t \begin{pmatrix} 9 \\ -13 \\ 9 \end{pmatrix}$, S(-4/9/-1)

26a) C_1(-1/1/2), C_2(-1/1/8) b) C(-1/1/30.75)

27a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 14 \end{pmatrix} + t \begin{pmatrix} 3 \\ -4 \\ -7 \end{pmatrix}$, $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ b) 31.32° c) $P(5\,^5/_7\,/\,5\,^5/_7\,/10)$

28. P(2/2/4), 90°

29. P(5/1/4) as well as D itself.

30. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$ resp. $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} 5 \\ 1 \\ -1 \end{pmatrix}$

7. The plane

35

1a) $3x + y - z - 6 = 0$
b) $2x - 3y + z - 12 = 0$
c) $3x - 4y + 5z - 6 = 0$
d) $4x + 21y - 28z - 105 = 0$
e) $34x - 7y + 67z - 232 = 0$
f) $135x - 95y + 164z - 836 = 0$

2a) $x - 2y - 3z - 4 = 0$
b) $5x - y + 2z - 4 = 0$

3. $3x + 2y + z - 7 = 0$

4a) $x + 7y - 4z + 19 = 0$
b) $6x - y + 2z - 20 = 0$

5a) $2x + z - 3 = 0$
b) $3y - 5z + 8 = 0$
c) $2x - 7y - 12 = 0$

6a) $x - 5 = 0$
b) $2y + 5 = 0$
c) $z - 8 = 0$

7a) $\mathcal{P} \parallel$ z-axis (1. projecting)
b) $\mathcal{P} \parallel$ x- axis (2. projecting)
c) $\mathcal{P} \parallel$ yz-plane (2. principal plane)
d) $\mathcal{P}$ through y- axis (3. projecting)

36

8a) $3x - 2y + z - 10 = 0$; S(0/-4/2) b) $8x + 12y - 3z - 16 = 0$; S(2/1/4)

9a) a = 2, b = - 6, c = 3
b) a = - 9, b = - 4, c = 12
c) a = - 20, c = 12
d) b = $^4/_7$, c = - 1

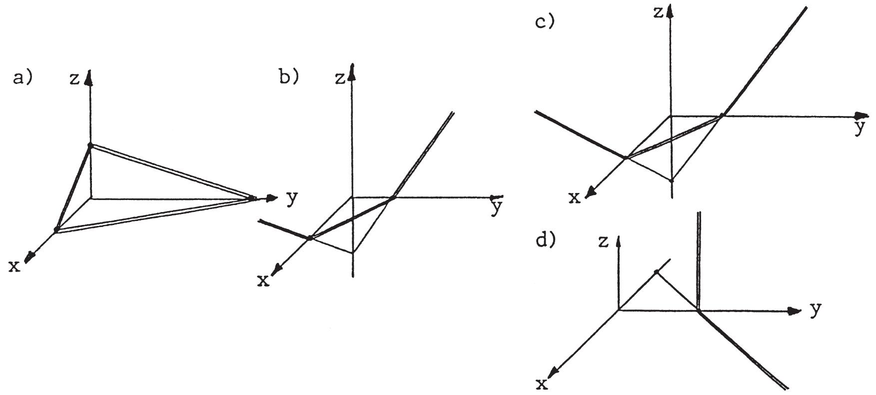
10a) a = 6, b = 12, c = 4
b) a = 10, b = 4, c = - 5
c) a = 9, b = 4 ½ , c = - 6
d) a = - 6, b = 5

Illustrations to exercise 10: on page 71

11. $bcx + acy + abz - abc = 0$ resp. $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ (axes intercept form for the equations for planes)

Illustrations to exercise 10

36



12a) $3x + 2y + 2z - 6 = 0$
b) $3x - 4y + 24z - 24 = 0$
c) $5x - 9y - 15z + 45 = 0$

d) $3x + 3y - 7z + 42 = 0$
e) $7x + 5y + 35 = 0$
f) $x - 7 = 0$

13a) $b = 6, c = -2$
b) $b_1 = 4.8, c_1 = 3;$
$b_2 = -3, c_2 = -4.8$

37

14a) $t_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix},$

$t_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 6 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix},$

$t_3: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

b) $t_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$

$t_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix},$

$t_3: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 8 \end{pmatrix} + t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

15a) $2x + 3y + 4z - 12 = 0$
b) $5x - 8y + 10z - 40 = 0$
c) $30x + 7y + 35z - 210 = 0$
d) $5x + 9z - 45 = 0$
e) $z - 3 = 0$
f) $4x + 10y + 15z - 60 = 0$
g) $3x + 4y + 12z - 60 = 0$
h) $x + 3y + 3z - 15 = 0$

16a) A yes, B no
b) A yes, B yes

17a) P(0/0/-4)
b) P(0.4/0.4/0.4)
c) P(-3/4/5)
d) P(1/1/7)

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18. yes ($\mathcal{P}$: $28x - 29y - 11z + 83 = 0$)
19. S(4/4/4)
20. $c = \pm 9$

21a) D(2/-1/2)
b) D(4/0/0)
c) $\ell \parallel \mathcal{P}$
d) D(3/3/3)

22a) D(0/3/-1)
b) D(1/-2/-3)
c) $\ell \in \mathcal{P}$
d) D(0/-1/-1)

39

23a) $3x + 2z - 12 = 0$
b) isosceles trapezoid, $A \approx 16.22$

39 24a) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ 0 \end{pmatrix} + t \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ b) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$

c) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 5 \\ -3 \\ 7 \end{pmatrix}$ d) ℓ: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t \begin{pmatrix} 4 \\ 1 \\ 2 \end{pmatrix}$

25. b = 3, c = 5

8. Normal forms

40 1a) $2x - y + 6z - 2 = 0$
b) $7x + 2y - 2z - 8 = 0$
c) $3x + z - 12 = 0$

2a) $x + 3y - 2z + 6 = 0$
b) $2x - 5z - 30 = 0$
c) $3x + 6y - 2z - 8 = 0$

3a) $4x - y - 3z + 18 = 0$
b) $5x - y - 4z + 32 = 0$

4. d = 11

5a) P(0/1/-4)   b) P(1/0/3)

6. ℓ_2: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -1 \\ 3 \end{pmatrix}$

7. D(1/-1/3); A = 27

41 8a) $3x - 4y - 10z + 9 = 0$
b) $29x - 38y - 7z - 51 = 0$
c) $x - 3 = 0$

9a) P'(4/1/-4)
b) P'(5/-2/1)
c) P'(11/-6/-15)

10. $4x + 3y - 9z + 10 = 0$

11a) ℓ': $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -2 \\ -4 \\ 5 \end{pmatrix}$ b) ℓ': $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 17 \\ 13 \end{pmatrix} + t \begin{pmatrix} 0 \\ 5 \\ 3 \end{pmatrix}$

c) ℓ': $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3 \\ -3 \\ -3 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ 7 \end{pmatrix}$ d) ℓ': $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -11 \\ 10 \\ -4 \end{pmatrix} + t \begin{pmatrix} 7 \\ -3 \\ -9 \end{pmatrix}$; $\ell \parallel \mathcal{P}$

12. R(2/-2/3)

13. $x + 4y - 3z + 3 = 0$

42 14a) ℓ': $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ -4 \end{pmatrix} + t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ b) P(2/2/6)

15a) 64.45°
b) 72.93°
c) 48.00°

16a) 7.66°
b) 0°; $\ell \parallel \mathcal{P}$

17. S(4/4/2); 36.60°

43 18a) B(4/0/0), D(2.4/-8/-3.2)
b) 24.09° ($\mathcal{P}$: $x - y + 2z - 4 = 0$)

19. $u = \pm 4$

20a) volume 108π
b) volume 243π

21. volume 45π (plane of the base circle: $10x + 2y - 11z + 5 = 0$; height 15, radius 3)

22a) $y = -7, z = -8$ b) $V_1(7/5.5/-11.5)$, $V_2(-1/-10.5/4.5)$ **43**

23a) $y_c = 16$, D(9/12/-4) b) $C'_1(2/7/9)$, $D'_1(9/3/5)$; $C'_2(-6/9/-7)$, $D'_2(1/5/-11)$

24a) $\beta = 90°$ b) D(1/5/-4) c) h = 36 d) $V_1(21/-26/0)$, $V_2(-11/38/8)$

44

25a) 2
b) 3
c) 1

26a) 4
b) 2.88
c) 3
d) 14.64

27a) 5
b) 7
c) 4

28a) 2
b) 63.43°

29. $4x - 4y - 7z - 21 = 0$; d = 3

45

30a) 4
b) 1
c) 2
d) 3

31a) $11x - 2y + 10z + 30 = 0$, $11x - 2y + 10z - 60 = 0$
b) $24x - 7z + 105 = 0$, $24x - 7z - 95 = 0$
c) $9x + 12y + 8z + 28 = 0$, $9x + 12y + 8z - 40 = 0$

32a) $6x + 2y - 8z + 13 = 0$, $2x - 6y - 7 = 0$
b) $21x - 4y + 11z + 7 = 0$, $9x - 16y - 23z + 11 = 0$
c) $10x + y - 7z - 35 = 0$, $10x - 23y + 11z + 13 = 0$
d) $11x + 22y - 5z - 32 = 0$, $x + 2y + 11z + 20 = 0$

33a) $P_1(-1/2/2)$, $P_2(5/-4/2)$
b) $P_1(3/1/4)$, $P_2(0/-1/6)$

34a) d = 2
b) $x_1 = 10$, $x_2 = -24$

35a) $P_1(0/0/4)$, $P_2(0/0/-3)$
b) P_2

36a) P(24/6/2)
b) d = 18

9. Circle and sphere

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1a) O(17/17), r = 17
b) O(5/7), r = 5
c) O(-20/13), r = 13
d) O(12/0), r = 15

2a) O(4/-3), r = 5
b) O(-1/-6), r = 6
c) O(-7/0), r = 8
d) O(3.5/4), r = 5.5
e) O(2.5/-1.5), r = 3
f) O($-\frac{2}{3}$/3), r = $\frac{1}{3}$

3a) $S_1(0/10)$, $S_2(-8/-6)$
b) P(-3/4)
c) $S_1(7/2)$, $S_2(0/5)$
d) $S_1(-11/65)$, $S_2(30/62)$

4a) $S_1(0/0)$, $S_2(-8/6)$; d = 5 (secant)
b) P(12/5); d = 13 (tangent)

5. c_1: $(x - 4)^2 + (y - 6)^2 = 25$; c_2: $(x + 1)^2 + (y - 11)^2 = 25$

6. c_1: $(x - 2)^2 + (y - 11)^2 = 25$; c_2: $(x + 6)^2 + (y - 17)^2 = 25$

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7a) $(x - 2)^2 + (y - 3)^2 = 0.64$
b) $(x - 3)^2 + (y - 6)^2 = 1$

8. $P_1(-3/2)$, $P_2(-7/-4)$

9. c_1: $(x - 49)^2 + (y + 17)^2 = 2601$
c_2: $(x + 41)^2 + (y - 31)^2 = 2601$

10a) $(x - 3)^2 + (y - 2)^2 = 26$
b) $(x - 4)^2 + (y + 1)^2 = 169$

11. $14x + 9y + 3 = 0$

12a) $12x + 5y - 44 = 0$
b) P(7/-8), Q(-3/16); $\overline{PQ} = 26$

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13. c_1: $(x - 9)^2 + (y - 6)^2 = 9$
c_2: $(x + 5)^2 + (y - 4)^2 = 9$

14. $P_1(11.2/-6.6)$, $P_2(-3.2/12.6)$;
$\overline{P_1P_2} = 24$

15. c_1: $(x + 9)^2 + (y + 5)^2 = 25$
c_2: $(x - 11)^2 + (y - 5)^2 = 225$

16. c_1: $(x - 4)^2 + (y + 1)^2 = 100$
c_2: $(x + 8)^2 + (y - 15)^2 = 100$

17a) $(x - 1)^2 + (y + 2)^2 = 169$
$(x + 39)^2 + (y - 38)^2 = 2809$
b) $(x - 3)^2 + (y - 4)^2 = 100$
$(x + 13)^2 + (y - 20)^2 = 676$

18a) $3x + 4y - 50 = 0$
b) $5x - 12y + 123 = 0$
c) $4x - 15y + 67 = 0$

48

19a) $4x + 3y - 20 = 0$, $4x + 3y + 30 = 0$
b) $7x - 24y + 639 = 0$, $7x - 24y - 611 = 0$

20a) $P_1(13/8)$, $P_2(-3/-4)$; $4x + 3y - 76 = 0$, $4x + 3y + 24 = 0$
b) $P_1(-8/7)$, $P_2(8/-23)$; $8x - 15y + 169 = 0$, $8x - 15y - 409 = 0$

21a) $x - 3y + 10 = 0$, $3x + y + 10 = 0$
b) gradients $^1/_3$ and -3.

22a) $P_1(3/5)$, $P_2(5/-3)$; $3x + 5y - 34 = 0$, $5x - 3y - 34 = 0$
b) $O(-2/6)$, $P_1(-4/12)$, $P_2(4/8)$; $x - 3y + 40 = 0$, $3x + y - 20 = 0$

23a) $(x - 3)^2 + (y + 1)^2 + (z - 2)^2 = 12$
b) $(x - 2)^2 + (y - 3)^2 + (z - 1)^2 = 19$

24a) $O(2/-1/5)$, $r = 2$
b) $O(-6/0/3)$, $r = 6$
c) $O(7/-2/0)$, $r = 0$
d) $O(0.5/-1.5/1)$, $r = 3$

25. $r = 21$

26a) $S_1(4/5/0)$, $S_2(6/-1/2)$
b) $S_1(5/2/0)$, $S_2(3/-4/2)$
c) $S_1(-1/1/6)$, $S_2(-6/-1/1)$
d) $P(2/2/0)$ (tangent)

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27a) $\overline{O_1O_2} = r_1 + r_2$
b) $P(7/0/-3)$

28a) $P(0/-1/-2)$
b) $P(-2/2/-4)$

29a) $(x - 6)^2 + (y - 5)^2 + (z+3)^2 = 4$
b) $(x+4)^2 + (y - 30)^2 + (z+5)^2 = 900$
c) $(x+2)^2 + (y+7)^2 + (z - 11)^2 = 144$

30a) $(x - 9)^2 + (y - 1)^2 + (z - 5)^2 = 256$
$(x - 9)^2 + (y - 1)^2 + (z - 5)^2 = 4$
b) $(x - 5)^2 + (y - 4)^2 + (z - 6)^2 = 144$
$(x - 5)^2 + (y - 4)^2 + (z - 6)^2 = 4$

31. $O(0/-3/0)$, $r = 12$

32. $O(6/7/8)$, $r = 12$

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33. S_1: $(x - 2)^2 + (y - 3)^2 + (z+4)^2 = 9$; S_2: $(x - 3)^2 + (y - 1)^2 + (z+5)^2 = 9$

34a) S_1: $(x - 11)^2 + (y - 12)^2 + (z - 9)^2 = 324$; S_2: $(x+5)^2 + (y+16)^2 + (z+7)^2 = 324$
b) S_1: $(x - 26)^2 + (y+37)^2 + z^2 = 1764$; S_2: $(x+6)^2 + (y - 39)^2 + (z - 16)^2 = 1764$

35a) $4y - 3z - 13 = 0$
b) $4y + 3z + 22 = 0$
c) $z = 2$; $4x - 7y - 4z + 17 = 0$
d) $y = -2$; $2x + 3y + 6z + 34 = 0$

36a) $3x + 2y - 6z - 72 = 0$,
$3x + 2y - 6z + 26 = 0$
b) $2x - 2y + z - 34 = 0$,
$2x - 2y + z + 20 = 0$

37a) $R(1/6/-5)$ b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 6 \\ -5 \end{pmatrix} + t \begin{pmatrix} -2 \\ 4 \\ -3 \end{pmatrix}$ c) 43.60°

38. $R(8/0/3)$, $S(4/2/4)$

39a) $(x - 1)^2 + (y - 1)^2 + (z - 1)^2 = 1$
b) $(x - {}^7/_3)^2 + (y - {}^7/_3)^2 + (z - {}^7/_3)^2 = {}^{49}/_9$

10. Cross product and scalar triple product

1. $\vec{a} \cdot \vec{c} = 0$ and $\vec{b} \cdot \vec{c} = 0$ 2. $\vec{c} = \begin{pmatrix} -25 \\ 1 \\ -20 \end{pmatrix}$; $\vec{a} \cdot \vec{c} = 0$ and $\vec{b} \cdot \vec{c} = 0$ **51**

3. follows from the definition

4a) $\vec{0}$ b) $\vec{a} = \vec{0}$ or $\vec{b} = \vec{0}$ or $\vec{a}$ and $\vec{b}$ collinear

5a) $\vec{0}$ b) $4(\vec{a} \times \vec{b})$ c) $\vec{0}$ d) $-7(\vec{a} \times \vec{b})$

6. $a = \sqrt{13}$, $b = \sqrt{14}$, $c = \sqrt{101}$; $\varphi \approx 48.15°$; $\sin\varphi = \frac{\sqrt{101}}{\sqrt{13} \cdot \sqrt{14}}$

7. $\vec{e}_z, -\vec{e}_y, \vec{e}_x, \vec{e}_y$ 8a) $\vec{u} = \vec{v} = \vec{w} = \begin{pmatrix} -28 \\ 24 \\ 24 \end{pmatrix}$ b) - c) u=v=w=44 **52**

9a) $3x - y + 2z - 4 = 0$ c) $2y - 3z + 6 = 0$
b) $8x - 3y + z = 0$ d) $4x + 3y + 2z + 1 = 0$

10a) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ 9 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 7 \\ 1 \end{pmatrix}$ c) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 4 \end{pmatrix} + t\begin{pmatrix} 5 \\ 2 \\ -7 \end{pmatrix}$

b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 7 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ d) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} + t\begin{pmatrix} -13 \\ 15 \\ -6 \end{pmatrix}$

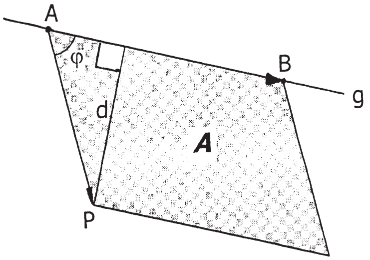
11. e.g. $\overrightarrow{AB}$ and $\overrightarrow{AC}$ collinear / C on line AB / $\overrightarrow{AB} \times \overrightarrow{AC} = \vec{0}$

12a) 5 b) 46 c) 38.5 d) 12

13a) $x + y + 2z - 1 = 0$ b) $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \\ -3 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$

14. $\frac{ab \cdot \sin\varphi}{ab \cdot \cos\varphi} = \tan\varphi$

15. $A = \overline{AB} \cdot d$
$d = \overline{AP} \cdot \sin\varphi$
$A = \overline{AB} \cdot \overline{AP} \cdot \sin\varphi$
$= |\overrightarrow{AB} \times \overrightarrow{AP}|$



16a) 3
b) 7 **53**
c) 11
d) 9

17. $A = 7.5$; $V = 50$

18a) $5\sqrt{2} \approx 7.07$ b) P(4/2/0)

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19. $\vec{d} = \vec{a} \times \vec{b}$

$h = |c \cdot \cos\varphi|$

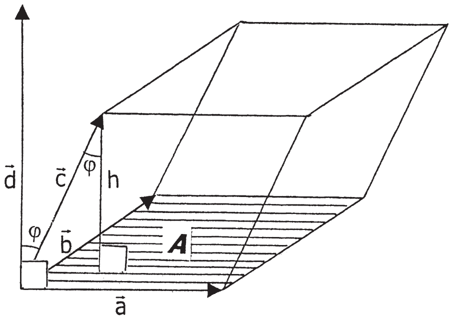
$d = |\vec{d}| = A$

$V = A \cdot h = d \cdot h$

$= |dc \cdot \cos\varphi|$

$= |\vec{d} \cdot \vec{c}|$

$= |(\vec{a} \times \vec{b}) \cdot \vec{c}|$



54

20a) 3
b) 78
c) 370
d) 53

21a) yes
b) yes
c) no
d) yes

22a) $x = -18$
b) $y = 3$
c) $x = 2$
d) $z = 2$

23a) yes
b) no
c) yes

24a) $x_1 = 6$, $x_2 = 8$
b) $y_1 = 5$, $y_2 = -8$
c) $z_1 = 1$, $z_2 = 1.8$

25. $V = \frac{1}{6} |(\overrightarrow{AB} \times \overrightarrow{AC}) \cdot \overrightarrow{AD}|$

26a) $V = 30$
b) $V = 15$
c) $V = 171$

27. $V = 3$

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28a) $\frac{1}{12}$
b) $\frac{1}{48}$

29a) S(1/1/1)
b) P(-1/3/0), Q(-2/1/-1), R(3/2/-2)
c) $V = 4.5$

30a) $x_1 = 0$, $x_2 = -22$
b) $y_1 = -1$, $y_2 = -7$

11. Exercises of vector geometry for the Matura-exam

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1a) O(4/5/1), $r = \sqrt{40} = 2\sqrt{10}$
b) 30°
c) $\ell: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 14 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$;
P(6/11/1)

2a) P(-2/2/2), $d = 6$
b) $(x + 4)^2 + (y - 4)^2 + (z - 3)^2 = 9$
c) $2x + y + 2z - 11 = 0$,
$2x + y + 2z + 7 = 0$

3a) C(-4/-11/1); $\alpha = \beta \approx 48.19°$, $\gamma \approx 83.62°$; area $A \approx 40.25$
b) C'(-8/-9/5); volume $V = 72$

4a) $(x - 7)^2 + (y - 8)^2 + (z - 7)^2 = 98$; point of contact P(0/8/0)
b) C(0/8/0) $\equiv$ P, D(3/9/-2); ABCD: $x + y + 2z - 8 = 0$; volume $V = 98$
c) O'(3.5/4.5/0), $r = 3.5\sqrt{2} \approx 4.95$

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5a) $(x+1)^2 + (y+8)^2 + (z - 4)^2 = 81$
b) $(x+1)^2 + (y+8)^2 + (z - 4)^2 = 9$
c) $(x - 1)^2 + (y+9)^2 + (z - 2)^2 = 36$; point of contact P(-3/-7/6)

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6a) $\overline{AB}=\overline{BC}=\sqrt{162}=9\sqrt{2}$ and $\overrightarrow{BA}\times\overrightarrow{BC}=0$

b) D(1/5/-4)

c) Height = 36; V_1 (21/-26/0), V_2 (-11/38/8)

7a) $\mathcal{P}$: x - y + 4z + 8 = 0 ; $\mathcal{S}$: $(x - 7)^2 + (y - 3)^2 + (z - 6)^2 = 72$

b) C(1+2t/9-2t/-t); show that $\overline{AC} = \overline{BC}$

c) $\overrightarrow{CA} \cdot \overrightarrow{CB} = 0$; or: intersect line ℓ with the sphere with center O(3/1/2) and radius r = 6; C(5/5/-2)

8a) $\mathcal{P}_2$: 2x - 2y - z + 6 = 0 ; identical normal vectors

b) d = 6

c) ℓ': $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 27 \\ 4 \\ 7 \end{pmatrix} + t \begin{pmatrix} 13 \\ 5 \\ 7 \end{pmatrix}$; S_1 (6/7/4), S_2 (-38/-21/-28)

d) $(x+3)^2 + (y+2)^2 + (z+5)^2 = 9$

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9a) O_1 (-3/7/0) , $r_1 = 10$

b) Point of contact C(3/-1/0)

c) $\mathcal{P}$: 2x - 2y + z - 8 = 0 ; $d = {}^{28}/_3$

d) $(x - 9)^2+(y+2)^2+z^2 = 25$

e) t: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix} + t \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$; tangent plane in P: 4x - 3y - 17 = 0

10a) $\vec{p} = \begin{pmatrix} 10 \\ -11 \\ -2 \end{pmatrix}$

b) c = 5 ; B(8/-10/5), C(18/0/0), D(8/11/2)

c) volume 450

11a) t_1: x = 0 ; t_2: 3x - 4y - 12 = 0

b) $(x - 5)^2+(y - 7)^2+(z - 12)^2 = 169$

c) 5x + 12z = 0

12a) P(6/5/4)

b) O_2 (4/7/3), $r_2 = 3$

c) point of contact C(2/9/2); tangent plane : 2x - 2y + z + 12 = 0